EXPLAINING THE VOLATILITY SMILE: 
REDUCED-FORM VS. STRUCTURAL OPTION MODELS*

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ABSTRACT

We employ a “reduced-form” approach to price European options. In contrast to “structural” models that assume stochastic processes for the underlying state variable(s), “reduced-form” models such as Stutzer (1996), Eberlein, Keller and Prause (1998), and Cochrane and Saa-Requejo (2000), directly fit the end distribution of the underlying state variable(s) with flexible statistical distributions. We derive an approximation formula that prices S&P 500 index options in closed form. Our model yields option prices that are more consistent with the data than the option prices that are generated by several widely used models. Although a quantitative comparison with other reduced-form models is more difficult, there are indications that our model is also more consistent with the data than these models.

Key words: reduced-form option pricing, S&P 500 index option

JEL classification: C14, C68, G12, G13
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1. INTRODUCTION

The literature demonstrates that the Black-Scholes (1973) model has moneyness
(volatility smile/skew) and maturity (volatility term structure) biases. Thus, researchers propose
various extensions to the Black-Scholes model that are more consistent with the data.\(^1\) Most of
these models assume that the rates of return on the underlying asset or its price follow some
stochastic process. We refer to such models as “structural models.” Bakshi, Cao and Chen (1997)
summarize and compare the empirical performances of some key variants of the Black-Scholes
model. They conclude that while these more advanced models do improve upon the Black-
Scholes model, there is a large gap between the option prices that are predicted by these models
and actual prices.

Some researchers believe that the disappointing empirical performances of the structural
models are due to the fact that they assume parametric structures that are too rigid to generate the
observed distribution of the price of the underlying asset. Thus, these researchers propose models
that obtain the distribution of the price of the underlying asset without assuming any stochastic
process.\(^2\) We refer to such models as “reduced-form models.” Stutzer (1996) uses a canonical
model and Eberlein, Keller, and Prause (1998) use a hyperbolic model to characterize the
distribution of the underlying asset. Considering an option contract as just another traded asset,
Hutchinson, Lo and Poggio (1994) adopt a neural network algorithm and Chidambaran, Lee, and
Trigueros (1999) apply a genetic algorithm to price the options. Cochrane, and Saa-Requejo
(2000) use the Sharpe ratio to price options and Shafer and Vovk (2001) bring game theory into

\(^1\) The extensions to the Black-Scholes model are too numerous to be reviewed here. We only highlight the
most relevant studies in this paper. Empirical research is briefly described in Section 4.

\(^2\) These models are different from the models that perfectly calibrate to the market option prices, such as,
for example, Rubinstein (1994), Ait-Sahalia and Lo (1995), and Jackwerth and Rubinstein (1996) that take
market prices as input.
the valuation of options. In this paper, which belongs to this broad category, we evaluate options within a standard asset pricing model and derive an approximated option pricing formula in closed form.³

Stock returns present substantial fat tails for short horizons but approach Gaussian as the holding period lengthens. Hence, parametric models often use jumps and random volatility with a fast decay to add weight to the tails of the distributions of short-term returns. However, these models require the estimation of a large number of parameters (for example, random volatility, random interest rate, and diffusion jump models require at least ten parameters) which may be cumbersome. Furthermore, the estimation requires the availability of prices of option contracts that differ only in their strike prices. However, frequently, the number of such option contracts at any time is not sufficient for the estimation of models with a large number of parameters, resulting in under-identification.

Our reduced-form model allows for the use of any distribution for the underlying asset. In particular the distribution need not be characterized or constrained by any parametric form. We then use the model in conjunction with the histogram of the returns on the underlying asset to price the options.⁴ Although the current reduced-form model is flexible, it yields a closed form approximation pricing formula and requires only very few parameters.

The empirical results using the S&P 500 index option contracts show that our model with only one calibration parameter substantially outperforms not only the original Black-Scholes (1973) model but also the six-parameter random-volatility jump-diffusion models of Scott (1997) and Bakshi, Cao and Chen (1997) and the four-parameter Heston (1993) model. There is also a substantial improvement of our model over the hyperbolic model. The trading exercises we

³ We follow previous “reduced-form” studies that, in contrast to traditional option pricing models, do not assume continuous trading.
⁴ A price histogram on the expiration date can be derived given the current price and a return histogram for the period between the current date and the expiration date. Thus, we use the histogram of returns till the expiration date and the histogram of prices on the expiration date interchangeably.
perform using our model further demonstrate the prediction power of our model.

The paper is organized as follows. Section 2 describes the model. Section 3 describes the empirical design where the first order approximation of the model is implemented. Section 4 describes data and the empirical results. Section 5 concludes.

2. MOTIVATION AND MODEL

Previous studies attempt to attribute the volatility smile/skew to the fact that the actual distribution of the returns of the underlying asset is more leptokurtic than a lognormal distribution. These studies focus mostly on short maturity option contracts because, as documented in the literature, the smile/skew dissipates as the option maturity lengthens [see, for example, Bakshi, Cao and Chen (1997), page 2022]. We observe that, correspondingly to the smile/skew dissipation, the fat tails in the distributions of the S&P 500 index shrink as the holding period lengthens. This is evident in the moments of the distribution of the returns on daily S&P 500 index from 1/3/1950 to 12/31/2007 for four arbitrary holding periods:

<table>
<thead>
<tr>
<th></th>
<th>1 month</th>
<th>3 months</th>
<th>6 months</th>
<th>12 months</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0880</td>
<td>0.0898</td>
<td>0.0902</td>
<td>0.0920</td>
</tr>
<tr>
<td>Std.Dev</td>
<td>0.1437</td>
<td>0.1431</td>
<td>0.1487</td>
<td>0.1562</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.4412</td>
<td>-0.3119</td>
<td>-0.1323</td>
<td>-0.1137</td>
</tr>
<tr>
<td>Extra Kurtosis</td>
<td>2.5524</td>
<td>1.5602</td>
<td>0.3447</td>
<td>-0.2048</td>
</tr>
</tbody>
</table>

Note that the absolute values of the skewness and kurtosis decrease substantially as the holding period lengthens (while the means and standard deviations are more or less stable). These decays in skewness and kurtosis are not easily captured in the existing models (see Das and Sundaram (1999) for details). Thus, we propose a model that extracts the ex-ante distribution of the returns of the underlying asset from the ex-post histogram of returns. Consider a generic classical asset pricing model with a stochastic kernel:
Eq. 1 \[ S_t = E_t[M_{t,r} S_t], \]

where \( M_{t,r} \) is the marginal rate of substitution, also known as the pricing kernel that discounts the future cash flow at time \( T \), \( E_t[\cdot] \) is the conditional expectation under the physical measure \( P \) taken at time \( t \), and \( S_t \) is the value of an arbitrary asset at time \( t \). The standard kernel pricing theory (e.g. Ingersoll (1989)) demonstrates that:

Eq. 2 \[ P_{t,T} = E_t[M_{t,r}], \]

where \( P_{t,T} \) is the risk free discount factor that gives the present value at time \( t \) of $1 paid at time \( T \).^5

The stock derivative, symbolized by \( C_t \), is priced similarly by Eq. 1, i.e. \( C_t = E_t[M_{t,r} C_t] \).

This results in the following theorem that, unlike its counterparts that are obtained by Black and Scholes (1973) and Brennan (1979), requires no assumptions on the distribution of returns or the preferences.

[Theorem] If the option and its underlying asset are perfectly (positively or negatively) correlated, then the following result holds:

Eq. 3 \[ C_t = P_{t,T} E_t[C_t] + b^5 \left\{ S_t - P_{t,T} E_t[S_t] \right\} \]

where \( b^5 = \frac{\text{cov}(C_t, S_t)}{\text{var}(S_t)} \) is known as the “dollar beta”.

The proof of the theorem is given in the Appendix. The intuition for Eq. 3 is as follows.

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^5 The usual separation theorem gives rise to the well-known, risk-neutral pricing result:
\[ S_t = E_t[M_{t,r} S_t] = E_t[M_{t,r}] E_t[S_t] = P_{t,T} E_t[S_t] \]

If the risk free interest rate is stochastic, then \( E_t[S_t] \) is the conditional expectation under the \( T \)-forward measure \( \tilde{P}^{T} \). When the risk free rate is non-stochastic, then the forward measure reduces to the risk neutral measure \( \tilde{P} \) and will not depend upon maturity time, i.e. \( E_t[S_t] \). Without loss of generality and for the ease of exposition, we shall assume non-stochastic interest rates and proceed with the risk neutral measure \( \tilde{P} \) for the rest of the paper.
The prices of two assets have a unity correlation when their values are linear functions of one another (with a slope that equals the ratio of standard deviations). The same holds for the cash flows on the expiration date. Let \( C_T = A S_T + B \). The present value of \( C_T \) for a risk-neutral investor equals \( A \) times the present value of \( S_T \) plus the present value of \( B \). The same holds for a risk-averse investor. However, \( B \) has the same present value for risk-neutral and risk-averse investors. Thus, in this case, the price discounts due to risk (i.e., differences between the present values of the cash flows for risk-neutral and risk-averse investors) for two assets are proportional to \( A \), the ratio of the standard deviations of their prices.

If the correlation is not unity, then we write the above equation as:

\[
C_t = P_{t,T} E_t[C_t] + b^t \{ S_t - P_{t,T} E_t[S_t] \} + \epsilon
\]

where

\[
e = (r_{MC}^s - r_{SC}^s) s_C^2 s_M^2
\]

and \( s_C^2 = \sqrt{\text{var}[C_T]} \), \( s_S^2 = \sqrt{\text{var}[S_t]} \) and \( s_M^2 = \sqrt{\text{var}[M_{t,T}]} \).

If the correlation between the option and its underlying stock is positive unity, then regardless of how they are correlated with the pricing kernel, \( r_{SM}^s = r_{CM}^s \) and \( r_{SC}^s = 1 \). Hence, \( \epsilon = 0 \). If the correlation between the option and its underlying stock is negative unity, then it must be that \( r_{SC}^s = -1 \) and \( r_{CM}^s = -r_{SM}^s \), which also implies that \( \epsilon = 0 \). If \( r_{SC}^s \neq 1 \), then \( \epsilon \) does not equal 0. In general, the semi-definite constraint gives the following result:

\[
r_{SM}^s r_{CM}^s - \sqrt{(1 - (r_{SM}^s)^2)(1 - (r_{CM}^s)^2)} \leq r_{SC}^s \leq +\sqrt{(1 - (r_{SM}^s)^2)(1 - (r_{CM}^s)^2)}
\]

We use a standard binomial model with \( n = 400 \) to gauge the magnitude of the errors. We examine various moneyness levels (20% in and out of the money), interest rates (3% to 9%), and volatilities (0.2 to 0.6). We find that the errors are relatively small and their magnitude is negatively related to moneyness (where higher moneyness levels represent more out of the
money), interest rates and volatilities. Detailed results are available upon request. The small magnitude of the errors motivates the use of Eq. 3 as an approximation of Eq. 4.

3. EMPIRICAL METHODOLOGY DESIGN

In this research, we propose an empirically based pricing model that is more flexible than the parametric models and has more predictive power than those models that take prices as given. As in Stutzer (1996), we assume that investors use the information from past return-realizations to estimate future returns on the underlying asset. In particular, we assume that, on each day investors construct histograms from a most recent fixed-length window.

The expected option payoff is calculated as the average payoff where all the realizations in the histogram are given equal weights. Thus, \( E_t[C_{T,T,K}] \) and \( E_t[S_i] \) are calculated as:

\[
E_t[C_{T,T,K}] = \frac{1}{N} \sum_{i=1}^{N} \max(S_{r,i} - K, 0),
\]

\[
E_t[S_i] = \frac{1}{N} \sum_{i=1}^{N} S_{r,i}
\]

where \( N \) is the total number of realized returns, \( S_{r,i} \) is the \( i \)-th element in the stock price histogram, and \( C_{T,T,K} \) is the price observed at time \( t \), of an option that expires at time \( T \) with strike price \( K \). Substituting the results in Eq. 7 in the approximation pricing formula of Eq. 3, we obtain our empirical model:

\[
C_{T,T,K}^{Our} = P_t \cdot E_t[C_{T,T,K}] + b^S \{ S_t - P_t \cdot E_t[S_i] \}
\]

\[
= P_t \cdot \frac{1}{N} \sum_{i=1}^{N} \max(S_{r,i} - K, 0) + b^S \left\{ S_t - P_t \cdot \frac{1}{N} \sum_{i=1}^{N} S_{r,i} \right\}
\]

where the dollar beta is defined as, \( b^S = \frac{\text{cov}(C,S)}{\text{var}(S)} \) as defined in Eq. 3.

We construct histograms from realizations of S&P 500 (SPX) returns. We calculate the price on day \( t \) of an option that settles on day \( T \) using a histogram of S&P 500 index returns for
a holding period of $T - t$, taken from a five-year window immediately preceding time $t$. For example, an \( x \)-calendar-day option price on any date is evaluated using a histogram of round[\( \frac{252}{365} x \)]-trading-day holding period returns where round[\( x \)] is rounding the nearest integer.\(^7\)

The index levels used to calculate these returns are taken from a window that starts on the 1260-th (\( \approx 5 \times 252 \)) trading day before the option trading date and ends one day before the trading date. Thus, this histogram contains 1260-round[\( \frac{252}{365} x \)]-trading-day return realizations.

Note that option prices should be based upon projected future volatility levels rather than historical estimates. We assume that investors believe that the distribution of index returns over the time to maturity follows the histogram of a particular horizon with a projected volatility. In practice, traders obtain this projected volatility by calibrating the model to the market price. We incorporate the projected volatility, \( \hat{\nu}_{t,T,K} \), into the histogram by adjusting it returns:

\[
R_{t,T,K,i} = \frac{\hat{\nu}_{t,T,K}}{\nu_{t,T}} (R_{t,T,j} - \bar{R}_{t,T}) + \bar{R}_{t,T}, \quad i = 1, \ldots, N,
\]

where the historical volatility \( \nu_{t,T} \) is calculated as the standard deviation of the historical returns as follows,

\[
\nu_{t,T}^2 = \frac{1}{N-1} \hat{\sigma}_{t,T}^2 = \frac{1}{N-1} \sum_{j=1}^{N} (R_{t,T,j} - \bar{R}_{t,T})^2
\]

where \( R_{t,T,j} = S_{t,j} / S_t \) and \( \bar{R}_{t,T} = \frac{1}{N} \sum_{j=1}^{N} R_{t,T,j} \) is the mean return.

Note that the transformation from \( R \) to \( \hat{R} \) changes the standard deviation from \( \nu_{t,T} \) to \( \hat{\nu}_{t,T,K} \), but does not change the mean, skewness, or kurtosis. The preservation of these moments is a constraint that we impose on our model so that the calibration of our model matches that of the Black-Scholes (1973) model. This is the only parameter we estimate in our model. We could easily further improve the calibration capability of our model by estimating additional parameters.

---

\(^6\) We use three alternative time windows, 2-year, 10-year and 30-year, to check the robustness of our procedure and results.

\(^7\) The conversion is needed because we use trading day intervals to identify the appropriate return histograms and calendar day intervals to calculate the appropriate discount factor.
(allowing more moments to be flexible). In the next section, we show that with just one parameter, our model already outperforms the complex six-parameter, jump-diffusion, random volatility model.

When $\nu_{t,T,K}$ is substituted for the historical standard deviation $\nu_{t,T}$, each stock price observation in the histogram is adjusted to be $S_{t,K,i} \circ R_{t,T,K} \circ S_i$ and then Eq. 7 becomes:

$$E_i[C_{t,T,K}] = \frac{1}{N} \hat{a} \sum_{i=1}^{N} \max\{S_{t,K,i} - K, 0\}$$

$$E_i[S_{t,K}] = \frac{1}{N} \hat{a} \sum_{i=1}^{N} S_{t,K,i}$$

Substituting the actual market price for the model price in Eq. 8, we arrive at:

$$C_{t,T,K}^{\text{Market}} = P_{t,T} E_i[C_{t,T,K}] + b^5 \left\{ S_t - P_{t,T} E_i[S_t] \right\}$$

which we use to solve for $\nu_{t,T,K}$. We compare the performance of our model with that of three parametric models: Black-Scholes (1973), Heston (1993), and Bakshi-Cao-Chen (1997). The Black-Scholes (1973) model assumes a log normal diffusion process for the underlying asset:

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t,$$

where $r$ is the expected rate of return on the underlying asset, $\sigma$ is the instantaneous standard deviation, and $dW_t$ represents the Wiener process whose differential has a zero mean and $dW_t$ variance. The Black-Scholes (1973) call option formula on the SPX is:

$$C_{t,T,K}^{\text{BS}} = S_t N(d_1) - e^{-r(T-t)} KN(d_2),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = d_1 - \sigma \sqrt{T-t}$$

We solve for the implied volatility of the Black-Scholes model, denoted as $\sigma^*$, by substituting the
market price of the call option into the pricing equation.

Several parametric models, especially those that incorporate random volatility and/or jumps, have been proposed in order to address the volatility smile/skew puzzle. Next, we compare our model to the Heston (1993) model. Let the volatility, \( s \), in the Black-Scholes (1973) model be random over time and \( \nu = s^2 \). Then, Heston lets it follow a mean reverting square root process. Hence, Eq. 13 is modified as follows,

\[
\text{Eq. 15: } \begin{cases}
    dS_t &= \mu S_t dt + \sqrt{\nu_t} dW_{1,t} \\
    d\nu_t &= \kappa (\theta - \nu_t) dt + \xi \sqrt{\nu_t} dW_{2,t}
\end{cases}
\]

where \( dW_{1,t} dW_{2,t} = \rho dt \). Note that conditional on a known volatility value, the stock price is still log normally distributed. Hence, Hull and White (1987) and Scott (1987) show that one can approximate the stochastic volatility option pricing model by simply replacing the “total variance” that is \( s^2 (T-t) \) in the Black-Scholes model by \( E_t [\nu (u)] du \). However, Heston (1993) shows that such a model has a closed form solution in the Fourier space:

\[
\text{Eq. 16: } C_{t,T,K}^{\text{Hest}} = S_0 P_1 e^{-r(T-t)} K P_2
\]

where

\[
\text{Eq. 17: } P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{iu \ln K} f_j}{iu} \right] du, \quad j = 1, 2
\]

\( i = \sqrt{-1} \) and \( f_j, j = 1, 2 \) is the characteristic function defined as:

\[
\text{Eq. 18: } f_j = e^{(\theta + D_j)\nu + i\alpha}
\]

where

\[
\text{Eq. 19: } \begin{cases}
    B_j &= riu(T-t) + \frac{\alpha}{\nu} (b_j - r\alpha u + d_j)(T-t) - 2\ln \left( \frac{\nu e^{\nu(T-t)}}{1 - g_j} \right) \\
    D_j &= b_j - r\alpha u + d_j \frac{\nu e^{\nu(T-t)}}{1 - g_j} - \frac{\nu e^{\nu(T-t)}}{1 - g_j}
\end{cases}
\]
\[ g_j = \frac{b_j - r a u + d_j}{b_j - r a u - d_j} \]
\[ d_j = \sqrt{(r a u - b_j)^2 - a^2(2 i u x_j - u_j^2)} \]

\( x_j = 1/2 \) and \( x_j = -1/2 \).

Bakshi-Cao-Chen (1997) as well as other similar models add to the model jumps that are independent of volatility and stock price processes. Hence, for each characteristic function, \( j = 1, 2 \) in Eq. 18, we multiply correspondingly the following characteristic function from the jumps:

**Eq. 21**

\[
\begin{align*}
\hat{f}_{j1}(u) &= \exp \left\{ i u l e^{m_j \gamma s_j^2} (T - t) \left( e^{v_j \gamma s_j^2} - 1 \right)^2 \right\} \\
&\quad + i u l e^{m_j \gamma s_j^2} (T - t) \left( e^{v_j \gamma s_j^2} - 1 \right)^2
\end{align*}
\]

and

**Eq. 22**

\[
\begin{align*}
\hat{f}_{j2}(u) &= \exp \left\{ i u l e^{m_j \gamma s_j^2} (T - t) \left( e^{v_j \gamma s_j^2} - 1 \right)^2 \right\} \\
&\quad + i u l e^{m_j \gamma s_j^2} (T - t) e^{m_j \gamma s_j^2} \left( e^{v_j \gamma s_j^2} - 1 \right)^2
\end{align*}
\]

where \( l, m, \) and \( s_j \) are the jump intensity, jump mean size and jump size standard deviation respectively. The full Bakshi-Cao-Chen (1997) model also has random interest rates. However, since the impact of random interest rates on option prices is minimal for short maturity option contracts, we choose not to use it in our comparison. This reduces the complexity of the model and the number of parameters that should be estimated.

The Heston (1993) model has four parameters for the random volatility process – initial volatility \( \nu_t \), volatility mean reversion \( a \), volatility reverting level \( b \), and volatility of volatility \( c \). The Bakshi-Cao-Chen (1997) model adds three parameters for the jump process: jump intensity \( l \), jump mean size \( m \), and jump size volatility \( s_j \). Option models for which more than
one parameter should be estimated do not allow an estimation of a unique value of the implied volatility for each option. Hence, for comparison between our model and the Heston and the Bakshi-Cao-Chen models, we choose to use all the option contracts on a given day to compute a single set of parameter values by minimizing the sum of squared errors of the predicted option prices.

We compare the total volatilities of the Black-Scholes (1973) model \( \sigma \sqrt{T - t} \), our model \( \mu_{t,r,k} \), the Heston (1993) model \( \sqrt{\mu_{t,r}^{\text{Hest}}} \), and the Bakshi-Cao-Chen (1997) model \( \sqrt{\mu_{t,r}^{\text{BCC}}} \). For the Heston model we calculate the expected instantaneous variance for an arbitrary future time \( u \) as:

\[
E[V_t] = V_t e^{\mu u} + \beta \left( 1 - e^{\mu u} \right)
\]

where \( \mu, \beta, \) and \( V_t \) are the estimated values using the market option data. Hence the expected total variance is just an integration:

\[
\mu_{t,r}^{\text{Hest}} = E \int_0^T V_t e^{\mu u} + \beta \left( 1 - e^{\mu u} \right) du = V_t e^{\mu u} + \beta \left( 1 - e^{\mu u} \right)
\]

For the Bakshi-Cao-Chen (1997) random volatility and jump diffusion model, the total variance is approximated by:8

\[
\mu_{t,r}^{\text{BCC}} = \mu_{t,r}^{\text{Hest}} + \sigma^2 + \epsilon^{1/2} \left( 1 - e^{\epsilon^{1/2}} \right) \eta
\]

4. DATA AND EMPIRICAL RESULTS

In this section we briefly discuss previous relevant empirical studies and present our data, empirical methods, and our results. We evaluate the performances of four models: the Black-Scholes (1973), the random volatility model of Heston (1993), the random-volatility jump-diffusion model of Bakshi, Cao and Chen (1997) and Scott (1997), and our reduced-form model.

8 See the Appendix for a more detailed derivation.
In this evaluation we examine three performance measures of these models: (1) whether the implied volatilities on each day (obtained only for the Black-Scholes model and for our model) are fixed across strike prices as predicted by the theory, (2) whether the model prices are close to actual prices, as measured by each model’s root mean square pricing error (RMSE) and (3) whether the models can detect pricing errors as indicated by the relationship between the implied volatility and eventual profit. The first criterion has been used frequently in previous studies. The second criterion allows a similar comparison of all the models, and complements the first as the sensitivity of the prices of option contracts to implied volatilities may vary across models. For this comparison we use the optimal set of parameters (i.e., the one that minimizes the RMSE) for each day under each model. For the third comparison we examine the relation between the payoffs to selling a naked option contract and the implied volatilities generated by the four models. Our findings indicate that our model prices S&P 500 call option more accurately than the other models.

In comparing the performance measures across models, we also examine, when allowed by data availability, the performance in three maturity-based sub-samples (7-39 days, referred to as short; 42-130 days, referred to as medium; and 133-221 days, referred to as long). We conclude that our model outperforms the other models in all the sub-samples. In terms of implied volatilities, the advantage of our model is largest for the short maturity sub-sample. This is consistent with the observation that the distribution of returns for short maturities (as compared to the corresponding distributions for longer maturities) is least similar to a normal distribution. However, because option values are positively related to maturities, the dollar RMSEs are also positively related to maturities. Consequently, the advantage of our model in terms of dollar RMSEs is larger for the medium maturity contracts than it is for the short maturity contracts.

Charles Cao has generously provided us with approximated prices of S&P 500 index call option contracts, matched levels of S&P 500 index, and approximated risk free 90-day T-Bill
rates for the period of June 1988 through December 1991.  For each day, the approximated option prices are calculated as the average of the last bid and ask quotes. Thus, given the high volume and liquidity of the S&P 500 option contracts, the prices of the various option contracts on each day are obtained from about the same time. This motivates estimating the same parameter value(s) for all the option contracts on a given day, but estimating different parameter values for option contracts on different days. Index returns are computed using daily closing levels for the S&P 500 index that are collected and confirmed using data obtained from Standard and Poor’s, CBOE, Yahoo and Bloomberg.

**A Brief Review of Relevant Studies**

The empirical literature on the Black-Scholes (1973) model is voluminous. Starting with Black (1975), the literature documents biases of the Black-Scholes model along two dimensions: moneyness and maturity. Subsequent studies continue to find similar biases regardless of whether they adjust or do not adjust for early exercise premiums (i.e., American style).

The literature consistently documents the strike price bias (i.e., the volatility smile/skew) and the time to maturity bias (i.e., volatility term structure) in various contracts. Rubinstein (1994) demonstrates that the implied volatility for S&P 500 index option is a sneer after the stock market crash in 1987. Shimko (1993) argues that the implied S&P 500 index distribution is

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9 The data are used in Bakshi, cao and chen (1997).
10 The (Ex-dividend) S&P 500 index we use is the index that serves as an underlying asset for the option. For option evaluation, realized returns of this index need not be adjusted for dividends unless the timing of the evaluated option contract is correlated with lumpy dividends. Because we use monthly observations, we think that such correlation is not a problem. Furthermore, in any case, this should not affect the comparison of the volatility smile between our model and the Black-Scholes model.
11 The first to document biases are Black and Scholes (1972) who find option prices for high (low) variance stocks to be lower (higher) than predicted by the model.
12 Examples that do not take the American premium into consideration include MacBeth and Merville (1979), Emanuel and MacBeth (1982), Rubinstein (1985), Geske et al. (1983), and Scott (1987). Whaley (1982) and Geske and Roll (1984) discuss possible biases if such premiums are not included. Examples that take into consideration of the American premium include Whaley (1986), who adopt the Geske-Roll-Whaley model, for American style S&P 500 futures options and Bodurtha and Courtadon (1987), who adopt the approximation algorithm by Mason (1979) and Parkinson (1977), for currency options.
13 Note that the Black-Scholes sneer found by Rubinstein (1994) is based upon data from one day.
negatively skewed and more leptokurtic than a lognormal distribution. Jackwerth and Rubinstein (1996) find that the S&P 500 index futures distribution before the crash of 1987 resembles the lognormal distribution, while the post-crash distribution exhibits leptokurtosis and negative skewness.\textsuperscript{14}

The smile (or skew/smirk) puzzle has been largely attributed to the normality assumption in the Black-Scholes (1973) model. The Black-Scholes model may bias option prices because it underestimates the tail probabilities of the return distribution. Several studies introduce jumps or stochastic volatility into the distribution of the underlying asset in an attempt to increase the weight of the tails of the return distribution. Bates (1996) (who uses the Deutsche Mark option contracts from 1984 through 1991) and Bakshi, Cao and Chen (1997) (who use S&P 500 index option contracts from 1988 through 1991) discover that the magnitude of the volatility smile/skew still exists for short dated option contracts even when the pricing model allows stochastic volatility and jumps. Duan’s (1996) GARCH model draws a similar conclusion. Das and Sundaram (1999) argue that jump-diffusion and stochastic volatility models do not satisfactorily resolve the smile/skew puzzle because these models do not generate the skew and extra kurtosis patterns that resemble reality. For example, the extra kurtosis generated by jump diffusion models (stochastic volatility models) declines with the holding horizon faster (more slowly) than in reality.\textsuperscript{15}

Recently, Carr and Wu (2004) and Huang and Wu (2005) provide a Time-Changed Lévy framework that nests all existing models. By adopting up to three jump processes and four stochastic volatility specifications, Huang and Wu test the out of money S&P 500 Index option

\textsuperscript{14} For additional evidence, see Bates (1991 and 1996) and Dumas, Fleming, and Whaley (1998).
\textsuperscript{15} In their Table 1, Das and Sundaram demonstrate that, using jump diffusion models, extra kurtosis for the three-month holding period is less than 8% of the extra kurtosis for the one-week holding period. In their Table 3, Das and Sundaram demonstrate that, using stochastic volatility models, extra kurtosis for the three-month holding period is more than 70% of the extra kurtosis for the one-week holding period. In contrast, the corresponding number during our sample period between January 3, 1950 and December 31, 2007 is 24%. Detailed calculations are available upon request.
contracts (both calls and puts). Their computed in-sample pricing errors show that out of the
money calls are mis-priced more than the out of the money puts.

Eberlein, Keller, and Prause (1998) find that assuming a hyperbolic function for the
distribution of returns on the underlying asset mitigates, but does not eliminate, the volatility
smile and the time-to-maturity effects. They suggest that options that are not at-the-money face
additional risk such as liquidity, and thus are more expensive. Longstaff (1995), Jackwerth and
Rubinstein (1996), Dumas et al. (1998), and Peña et al. (1999) report that transaction costs and
liquidity contribute to, but do not completely explain, the volatility smile/skew.

**Implied Volatilities and Prediction Errors: Levels and Variability**

First, we plot the implied volatilities that are obtained for each option contract from the
Black-Scholes (1973) model and from our model for the three maturity-based sub-samples. We
present the annualized implied volatilities in Figure 1a through Figure 1c. Recall that an option
pricing model may generate an implied volatility smile/skew either because the model
underprices in- and out-of-the-money options, or because the market overprices these option
contracts. If the market does not overprice in- and out-of-the-money option contracts, then our
flat fitted implied volatilities and the volatility smile/skew under the Black-Scholes model
indicate that the Black-Scholes model (but not our model) overprices in- and out-of-the-money
option contracts.\footnote{To further examine whether a market overpricing of in- and out-of-the-money option contracts generates the non-flat pattern of implied volatilities, we calculate the payoffs generated by selling naked option contracts and examine the relationship between the payoffs and the implied volatilities.}

The summary statistics of the implied volatilities are provided in Table 2. Panel A of
Table 2 exhibits the statistics for the entire sample, and Panel B presents the statistics for the three
maturity sub-samples. Panel A indicates that the averages of the implied volatilities under the
Black-Scholes (1973) (0.2085) model and under our model (0.2013) are similar to one another.
However, the range of the implied volatilities under the Black-Scholes model is about 80% (from
8.18% to over 89%) while under our model the range is only about 55% (from 6.24% to 61.71%). Similarly, the standard deviation and the coefficient of variation of our implied volatilities (4.59% and 0.228, respectively) are smaller than those obtained from the Black-Scholes model (7.53% and 0.3612, respectively). Panel B indicates that, under the Black-Scholes model, the standard deviation and the coefficient of variation (about 11% and 0.47, respectively) of the implied volatilities for the short maturity option contracts are almost twice as large as the corresponding measures for the long and medium maturity option contracts (where they are about 40%-49%, 4.5%-5.9%, and 0.23 – 0.29, respectively). In contrast, our implied volatilities are relatively stable across the three maturity-based sub-samples. The standard deviation and the coefficient of variation for the short maturity option contracts are less than 50% larger than the corresponding measures for the long and medium maturity option contracts. These observations are consistent with a severe smile/skew of implied volatility under the Black-Scholes model, especially for short maturities, and with the absence of a severe smile/skew under our model. Comparing the implied volatilities that we obtain from our model and from the Black-Scholes model across the three maturity-based sub-samples, it seems that the advantage of our model over the Black-Scholes model is largest for the short maturity sub-sample. The coefficient of variation of the implied volatilities that are associated with the short maturity option contracts under the Black-Scholes model is 73% larger than the corresponding coefficient of variation under our model. For the medium and long maturity option contracts the corresponding figures are 33% and 20% respectively. This is consistent with the visual evidence presented in Figure 1a through Figure 1c.

Next we compare the pricing performance of the four models we evaluate: the Black-Scholes (1973), the random volatility model of Heston (1993), the random-volatility jump-diffusion model of Bakshi, Cao and Chen (1997) and Scott (1997), and our reduced-form model. Recall that for the random volatility model and the random-volatility jump-diffusion model we cannot estimate a unique set of parameters based on one contract. Recall also that the parameters
should be similar for contracts with different strike price that are priced on the same day, but may differ across trading days. Thus, for each trading day and each model, we find the parameter (or set of parameters) that minimizes the sum of squares of the pricing errors for all the option contracts that are traded on that day. We estimate a daily parameter, volatility, for the Black-Scholes model and for our model. For the Heston model we estimate four daily parameters (initial volatility value and three parameters in the volatility process). For the random volatility jump diffusion model we estimate six parameters (adding the jump size mean and jump size volatility to the four Heston model parameters). The larger is the number of the estimated parameters in a model, the better should be its ability to fit multiple option prices. Furthermore, the Black-Scholes model, the Heston model, and the Bakshi-Cao-Chen model are nested. Hence, we should expect the pricing errors in these models to be negatively related to the number of parameters.

We then calculate, for each model and each trading day, the pricing errors for each contract and the RMSEs (Root Mean Squared Errors) of these daily errors according to each model. We proceed to calculate the average, median and standard deviation of the RMSEs. The results are presented in Table 3. Panel A presents estimates of the magnitudes of the RMSEs when the parameters for each day are estimated from the option prices on that day (“in-sample test”). The averages of the RMSEs are $0.99 for the Black-Scholes (1973) model, $0.79 for the Heston (1993) model, and $0.78 for the random-volatility jump-diffusion model. The average RMSE for our model is $0.52. This result is rather remarkable as our one-parameter model substantially outperforms the six-parameter random-volatility jump-diffusion model and the four-parameter Heston model. Furthermore, we note that our model has also the smallest median and standard deviation of its RMSEs, indicating that our model consistently generates smaller RMSEs, and that the differences in means are not due to a few extreme outcomes.

Panel B presents the corresponding statistics when the parameters that are used on each
day are estimated from the option prices on the previous day (“out-of-sample test”). The ranking of the RMSEs is unchanged, and the mean RMSE from our model is still $0.26 smaller than the mean RMSE from the next best model (the Bakshi-Cao-Chen model). As expected, the RMSEs from the out-of-sample tests are larger than those from the in-sample tests. The difference using our model is $0.10, while the differences using the Black-Scholes, Heston and Bakshi-Cao-Chen models are $0.04 to $0.10. We conclude that the advantage of our model is robust to using the in-sample or out-of-sample test.

Panels Panel C, D and E of Table 3 present the corresponding summary statistics (averages, medians and standard deviations) of the RMSEs (using in-sample tests) for the short, medium and long maturities sub-samples, respectively. In these tests, the models are re-estimated for each maturity sub-sample. It should be noted that for these sub-samples we have fewer observations in each day and hence we do not have enough option contracts to estimate the random-volatility jump-diffusion model or even the Heston (1993) model parameters for all the days. We note that the advantage of our model is even larger when the models are estimated separately for each maturity sub-sample. Recall that our RMSEs from the entire sample (reported in Panel A) are about half as large as those from the Black-Scholes model, and about two thirds of those from the Heston and Bakshi-Cao-Chen models. Our RMSEs when the models are estimated separately for each maturity sub-sample (reported in Panels C, D and E) are less than one quarter of the Black-Scholes RMSEs, and less than one half of the Heston and Bakshi-Cao-Chen RMSEs. We conclude that our model outperforms the three other models in all three maturity sub-samples.

Comparing the four models across sub-samples we conclude that, in terms of dollar RMSEs, the advantage of our model is largest for the short and medium maturity sub-samples. At the first glance, this result seems to be inconsistent with the evidence in Figure 1 and Table 2 (that our model outperforms the Black-Scholes (1973) model most in the short maturity sub-sample).
The apparent difference between the implications of the results reported in Table 3 vs. those in Figure 1 and Table 2 may be due to the distinction between measuring the performance in dollar differences (in Table 3) vs. implied volatilities (in Figure 1 and Table 2). This difference may be due to the positive relation between option maturities and the sensitivity of their values to implied volatilities. Thus, although the advantage of our model in terms of implied volatilities for medium-maturity option contracts is smaller than for short-maturity option contracts, the corresponding advantages in dollar RMSEs for medium-maturity option contracts is larger than for short-maturity option contracts. The advantage of our model for the medium maturity sub-sample may exceed the advantage for the long maturity sub-sample because the fat-tails problem for the long maturity sub-sample is not as substantial as it is for the shorter maturity sub-samples.

Table 4 presents, for each pair of model, the frequencies that the RMSEs of one model are smaller than the RMSEs of the other model. It also presents the average difference in RMSEs in each case. Model A should be considered better than model B if it outperforms model B (i.e., has smaller RMSEs) more frequently than being outperformed by it, and if it does so by larger margins than those when it is outperformed. Again, Panel A of Table 4 reports results for the entire sample using an in-sample test. Unless two models are nested (e.g. Bakshi-Cao-Chen (1997) nests Heston (1993) that in turn nests Black-Scholes (1973)), no model is better than another model for all the option contracts. However, according to the above criterion, our model is better than all the other models that we consider. Our model outperforms the Black-Scholes, Heston and Jump models in 84 percent, 72 percent, and 72 percent of the days respectively. The corresponding margins when our model outperforms these models are, respectively, $1.13, $0.79 and $0.79, while margins when our model is outperformed by these models are, respectively, $0.23, $0.28 and $0.28.

Panel B reports the corresponding statistics using an out-of-sample test. Compared to the results that are reported in Panel A, the results in Panel B indicate that our model outperforms the
Heston and Bakshi-Cao-Chen models by slightly larger margins, and the Black-Scholes by a slightly smaller margin when the estimates are obtained from out-of-sample tests. Panels C, D and E report the results for the three maturity sub-samples. In all three sub-samples our model outperforms the three other models. We conclude that the advantage of our model is robust to using the in-sample or out-of sample test, and to the maturity of the option.

**Implied Volatilities and Profit Prediction Errors**

In this subsection, we compare the models by their prediction powers for profits from trading option contracts. Assume an investor who sells a call option and holds it till maturity. The profit from trading an option contract is computed as the initial option price minus the discounted (at the risk free rate) payoff on the expiration date. We investigate how the profit from selling each contract is related to the contract’s implied volatilities that are obtained from the various models. According to any model, the implied volatilities may depend on the time a price is quoted and on the expiration date, but not on the option’s moneyness. Thus, differences in implied volatilities of option contracts that differ only in their moneyness should be due to either pricing errors or an inappropriate model. A positive association between implied volatilities and trading profits is consistent with the hypothesis that the differences in implied volatilities are (at least partly) due to pricing errors, and that the model is capable of identifying these errors. A negative or insignificant association may indicate that the model’s sensitivity to moneyness does not match reality well.

We estimate the following seven regressions and report the results in Table 5:
In these regressions, $b_1$ through $b_4$ are the coefficients of the implied volatilities from the four models that we examine (i.e., the Black-Scholes (1973), our, the Heston (1993), and the Bakshi-Cao-Chen (1997) models), respectively. The variable $r$ is the realized index return for the period between the date on which the option price is observed and the date on which the option expires. We include this variable, which in efficient markets cannot be correlated with option prices, in order to control for the impact on profits of events that occur after the option is priced. We expect its coefficient to be negative because increases in the price of the underlying asset hurt the seller of a call option. The variable $\ln[T - t]$ is introduced to adjust for the use of the risk-free rate (rather than a risk-adjusted $r^*$) as the discount rate. The use of the risk-free rate reduces the calculated profits by $(\text{expression})$. This expression increases in $T - t$ (at a decreasing rate and thus we use $\ln[T - t]$ as a proxy). We expect the coefficient of this variable to be positive. The variable $D$ is the dollar difference between the market price of the option (which equals the Black-Scholes model price with the implied volatility generated by the Black-Scholes model) and the Black-Scholes model price with the implied volatility generated by our

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$\text{Eq. 26}$

\[ a. \quad P = a + b_1 s^* + b_2 r + b_3 \ln[T - t] + e, \]

\[ b. \quad P = a + b_1 \frac{\hat{V}}{\sqrt{T - t}} + b_2 r + b_3 \ln[T - t] + e, \]

\[ c. \quad P = a + b_1 \sqrt{\frac{V_{\text{Has}}}{T - t}} + b_2 r + b_3 \ln[T - t] + e, \]

\[ d. \quad P = a + b_1 \sqrt{\frac{V_{\text{BCC}}}{T - t}} + b_2 r + b_3 \ln[T - t] + e, \]

\[ e. \quad P = a + b_1 s^* + b_2 \sqrt{\frac{\hat{V}}{T - t}} + b_3 r + b_3 \ln[T - t] + e, \]

\[ f. \quad P = a + b_1 s^* + b_2 \sqrt{\frac{\hat{V}}{T - t}} + b_3 \sqrt{\frac{V_{\text{Has}}}{T - t}} + b_3 \sqrt{\frac{V_{\text{BCC}}}{T - t}} + b_4 r + b_3 \ln[T - t] + e, \]

\[ g. \quad P = a + b_1 r + b_3 \ln[T - t] + b_4 D + e. \]
model. We use this variable because implied volatilities are not linear functions of pricing errors.

The estimates of the first four regressions that are reported in Table 5 indicate that only the implied volatilities that are obtained from our model are significantly associated with profits from selling option contracts with different moneyness levels in the direction that is consistent with a well-specified model. The (negative and significantly different from zero) coefficient of the implied volatilities that are obtained using the Black-Scholes (1973) model has the wrong sign. The coefficients of the implied volatilities that are generated by the Heston (1993) and Bakshi-Cao-Chen (1997) models are also negative, but are not statistically different from zero.

While each of the first four regressions includes the implied volatilities that are obtained from one model, the fifth and sixth regressions include the implied volatilities that are obtained from our model as well as those that are obtained from other models (the Black Scholes model in the fifth and all models in the sixth). In the last regression the variable $D$ replaces the implied volatilities. In the sixth regression, the coefficient of the implied volatilities that are obtained from the Heston (1993) model is negative and significantly different from zero, while the coefficient of the implied volatilities that are obtained from the Bakshi-Cao-Chen (1997) model is positive and significantly different from zero (but not as large as the coefficient of the implied volatility that is obtained from our model). The other coefficients are very similar to corresponding coefficients in the first five regressions.

We further examine whether the superior performance of our model is related to option maturity by running similar regressions for each of the three maturity sub-samples.

Corresponding to our estimates of Eq. 26, we run the following regressions:

Eq. 27

\[
\begin{align*}
    a. \quad P &= a + b_1 s \sqrt{T - t} + b_2 r + b_3 \ln(T - t) + e, \\
    b. \quad P &= a + b_1 \tilde{v} + b_2 r + b_3 \ln(T - t) + e, \\
    c. \quad P &= a + b_1 s \sqrt{T - t} + b_2 r + b_3 \ln(T - t) + e, \\
    d. \quad P &= a + b_2 r + b_3 \ln(T - t) + b_4 D + e.
\end{align*}
\]
Table 6 reports the estimates we obtain from each of the four regressions using maturity sub-samples. In all the three sub-samples, the coefficients of the implied volatilities that are obtained from the Black-Scholes (1973) model are negative and significantly different from zero, while the coefficients of the implied volatilities that are obtained from our model are positive and significantly different from zero. Thus, we conclude that our model can identify mis-pricing errors better than the Black-Scholes model for all contract maturities.

Comparison with Other Reduced-form Models

We cannot directly compare our model to other reduced-form models because their calculations are not described in the studies that present them in sufficient detail. Nevertheless, we can still gain insight into a comparison of the performance of our model and other reduced-form models by what is reported in the other studies. Stutzer (1996) provides no empirical evidence of the performance of the canonical model. However, in his simulations, he finds that (see his Table 2) the implied volatility curve of the Black-Scholes (1973) model when it is calibrated to the canonical prices is a mild smile for short maturities (one month), an upward skew and a mild downward skew for medium maturities (three and six months), but an inverted smile for long maturities (one year).

We cannot accurately compare our results with the results by Eberlein, Keller, and Prause (1998) because of two reasons. First, we compute root mean square errors and they compute absolute mean errors. Second, we do not use the same sample. However, we can obtain some indication from their Tables 2 and 3 and our Table 3. Our Table 3 indicates that, for the entire sample, the daily Root Mean Square Errors (RMSEs) of our model’s option prices are about $0.40-$0.50, and are 40%-50% smaller than the corresponding RMSEs that are obtained from the Black-Scholes model. In contrast, Table 3 in Eberlein et al. indicates that the difference between the weighted means of their absolute pricing errors and the corresponding pricing errors from the Black-Scholes model is around $0.01-$0.02, and their Table 2 indicates that these differences are
small (less than 5%) relative to the pricing errors (noting that the average absolute pricing errors are larger than the average pricing error that are reported in Table 2). Thus, we think that our model improves upon the Black-Scholes model by a substantially larger margin than the improvement of the Eberlein et al. model.

Neural network and genetic algorithm performances are understood to be worse than parametric models such as Heston (1993) or Bakshi-Cao-Chen (1997).\(^ {19} \) The result in this section demonstrates that our model should outperform the neural network or genetic models because our model outperforms the parametric models.

5. CONCLUSION

We derive and test a “reduced-form” empirically based equilibrium option pricing model for S&P 500 index options. Following previous “reduced-form” models, we do not specify any stochastic process that the underlying asset follows. The theoretical model is based upon a model-free result derived in the paper in which we do not assume a parametric process and do not use any smoothing procedures. In contrast to previous reduced-form models, our model has a closed form solution for an approximation of the option price and is intuitive and easy to implement. The empirical model uses the histogram of the S&P 500 index returns to price options on this index. The success of our model relative to previous models indicates that option prices reflect the fat tails in the actual distribution of the underlying S&P 500 returns. Our model is the first to be consistent with the actual distribution of the underlying S&P 500 returns (as opposed to imposing arbitrary structure on this distribution).

Our one-parameter model outperforms the complex jump-diffusion-random-volatility model that needs six parameters. Head-to-head comparison of our model against the Black-Scholes (1973) model reveals that our model provides a much better prediction power. Our

\(^ {19} \) See Wu (2000) for the review and comparison.
regressions indicate that profits from selling naked option contracts are positively related to our implied volatilities but are negatively related to the Black-Scholes’ implied volatilities. These regressions control for the impact of ex-post realized returns, and take into account the non-linear relationship between implied volatilities and option premiums. Finally, our results are robust to a number of alternative empirical model specifications including various time horizons for the histogram and sample choices.

Our findings are consistent with the view that our model is more appropriate than complex parametric models to value S&P 500 call options. Furthermore, they also imply that the Black-Scholes (1973) model underprices in- and out-of the money call options relative to at-the-money options. Thus, in order to equate the option price to the actual price, the Black Scholes model generates relatively high implied volatilities for in- and out-of the money options relative to at-the-money options.

Our model is likely to perform also better than other reduced-form models. While direct comparisons between our model and other reduced-form models are not possible because their calculations are not described in the studies that present them in sufficient detail, casual comparisons imply that our model outperforms other reduced-form models.
APPENDIX

Variance under Jump Diffusion

In a jump-diffusion case with a jump size \( Y \) and an intensity parameter \( \lambda \), we have the following distribution:

\[
\ln S = \begin{cases} 
\ln S_0 + mt - \frac{\lambda}{2} \int_0^t V_u du + \sqrt{V_u} dW_u & \text{no jump} \\
\ln S_0 + \ln Y + mt - \frac{\lambda}{2} \int_0^t V_u du + \sqrt{V_u} dW_u & \text{jump}
\end{cases}
\]

where \( V \) and \( Y \) are random. The mean

\[
E[\ln S] = e^{\frac{1}{2}mt}E[\ln S_0] + (1 - e^{\frac{1}{2}mt})E[\ln S_0] \\
= \ln S_0 + mt - \frac{\lambda}{2} \int_0^t V_u du + (1 - e^{\frac{1}{2}mt})E[\ln Y]
\]

and variance

\[
\text{var}[\ln S] = E[\ln S^2] - E[\ln S]^2 \\
= \left\{ e^{\frac{1}{2}mt}E[\ln S_0^2] + (1 - e^{\frac{1}{2}mt})E[\ln S_0^2] \right\} - \left\{ e^{\frac{1}{2}mt}E[\ln S_0] + (1 - e^{\frac{1}{2}mt})E[\ln S_0] \right\}^2 \\
= e^{\frac{1}{2}mt} \text{var}[\ln S_0] + (1 - e^{\frac{1}{2}mt}) \text{var}[\ln S_0] + e^{\frac{1}{2}mt}(1 - e^{\frac{1}{2}mt}) \{ E[\ln S_0] - E[\ln S_0^2] \} \\
\text{var}[\ln S_0] + (1 - e^{\frac{1}{2}mt}) \text{var}[\ln Y] + (1 - e^{\frac{1}{2}mt})E[\ln Y] \\
\mu_0 \int_0^t V_u du + (1 - e^{\frac{1}{2}mt})(\mu^2 + \sigma^2)
\]

Proof of Theorem

By Eq. 1, the option price must follow \( C_t = E_t[M_t, C_T] \), and hence:

\[
C_t = E_t[M_t, C_T] \\
\text{Eq. 28} = E_t[M_t, E_t[C_T]] + \text{cov}[M_t, C_T] \\
= P_{t,T} E_t[C_T] + \text{cov}[M_t, C_T]
\]

Further expand the covariance term:
\[ \text{cov}[M_t, C_T] = r_{MC} s_M s_C = (\text{sgn}[r_{SC}] v_{MS} + e_1) s_M s_C \]
\[ = \text{sgn}[r_{SC}] v_{MS} s_M s_C + e_1 s_M s_C \]
\[ = \text{sgn}[r_{SC}] v_{MS} s_M s_C \frac{s_C}{s_S} + e_1 s_M s_C \]
\[ = \text{sgn}[r_{SC}] \text{cov}[M_t, S_T] \frac{s_C}{s_S} + e_1 s_M s_C \]
\[ = b \text{cov}[M_t, S_T] + e_1 s_M s_C + e_2 \]

where \( s_c = \sqrt{\text{var}[C_T]} \), \( s_s = \sqrt{\text{var}[S_T]} \) and

\[ e_2 = \left(\frac{\text{sgn}[r_{SC}] s_C}{s_S} - b\right) \text{cov}[M_t, S_T] \]

Given perfection correlation, \( b = \frac{\text{cov}[S_C]}{\text{var}[S_T]} = \text{sgn}[r_{SC}] \frac{s_C}{s_S} \). Eq. 29 becomes:

\[ \text{cov}[M_t, C_T] = b \text{cov}[M_t, S_T] + e_1 s_M s_C \]

Also, under perfect correlation, \( r_{MC} = \text{sgn}[r_{SC}] v_{MS} \) and consequently \( e_1 = 0 \). We can then further simplify Eq. 30 to:

\[ \text{cov}[M_t, C_T] = b \text{cov}[M_t, S_T] \]
\[ = b(E[M_t, S_T] - E[M_t] E[S_T]) \]
\[ = b(S_T \cdot P_t E[S_T]) \]

Substituting this result back into Eq. 28 and verifying that \( e_1 + e_2 = e \) complete the proof.
REFERENCES


Table 1
Summary Statistics of the Call Options

The number of the CBOE S&P 500 call option contracts in four moneyness-maturity groups. The trading day for all option contracts is not earlier than June 1988 and not later than December 1991. The moneyness is defined as \((S-K)/K\). The short (7 ~ 39 days), medium (42 ~ 130), and long (133 ~ 221 days) maturities refer to the first, second and third maturity dates in the data set.

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Short</th>
<th>Medium</th>
<th>Long</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt;5% in-the-money</td>
<td>3,609</td>
<td>3,514</td>
<td>4,184</td>
<td>11,307</td>
</tr>
<tr>
<td>0~5% in-the-money</td>
<td>2,628</td>
<td>2,499</td>
<td>2,308</td>
<td>7,435</td>
</tr>
<tr>
<td>0~5% out-of-the-money</td>
<td>2,655</td>
<td>3,054</td>
<td>2,908</td>
<td>8,617</td>
</tr>
<tr>
<td>&gt;5% out-of-the-money</td>
<td>307</td>
<td>1,598</td>
<td>3,253</td>
<td>5,158</td>
</tr>
<tr>
<td>Total</td>
<td>9,199</td>
<td>10,665</td>
<td>12,653</td>
<td>32,517</td>
</tr>
</tbody>
</table>
Table 2
Summaries of Implied Volatilities

This table summarizes the distribution statistics of the implied volatilities from the Black-Scholes (1973) model and our model. Panel A presents the whole sample and Panel B breaks down by maturity sub-samples. The scatter plots of the volatilities are given in Figure 1.

**Panel A (Whole Sample)**

<table>
<thead>
<tr>
<th></th>
<th>BS</th>
<th>Our</th>
</tr>
</thead>
<tbody>
<tr>
<td>min</td>
<td>0.0818</td>
<td>0.0624</td>
</tr>
<tr>
<td>max</td>
<td>0.8914</td>
<td>0.6171</td>
</tr>
<tr>
<td>mean</td>
<td>0.2085</td>
<td>0.2013</td>
</tr>
<tr>
<td>std.Dev</td>
<td>0.0753</td>
<td>0.0459</td>
</tr>
<tr>
<td>Coef. of Var.</td>
<td>0.3612</td>
<td>0.2280</td>
</tr>
</tbody>
</table>

**Panel B**

<table>
<thead>
<tr>
<th></th>
<th>Short</th>
<th>Medium</th>
<th>Long</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BS</td>
<td>Our</td>
<td>BS</td>
</tr>
<tr>
<td>Min</td>
<td>0.0818</td>
<td>0.0624</td>
<td>0.1069</td>
</tr>
<tr>
<td>Max</td>
<td>0.8914</td>
<td>0.6171</td>
<td>0.5948</td>
</tr>
<tr>
<td>mean</td>
<td>0.2364</td>
<td>0.2082</td>
<td>0.5948</td>
</tr>
<tr>
<td>std.Dev</td>
<td>0.1106</td>
<td>0.0563</td>
<td>0.0586</td>
</tr>
<tr>
<td>Coef. of Var.</td>
<td>0.4679</td>
<td>0.2704</td>
<td>0.2920</td>
</tr>
</tbody>
</table>
Table 3
Summary Statistics for Root Mean Square Errors

Panel A presents the average, median and standard deviation of the daily Root Mean Square Errors (RMSEs) that are calculated from the prediction errors for the prices of the option contracts under the parameter set that minimizes the RMSE for each day and model (in-sample test). Panel B presents these statistics when the parameters minimize the RMSE for the previous day (out-of-sample test). Panel C, D and E report the corresponding statistics from in-sample tests for the three maturity sub-samples [short (7 ~ 39 days), medium (42 ~ 130), and long (133 ~ 221 days)], respectively.

Panel A – In Sample

<table>
<thead>
<tr>
<th>RMSE</th>
<th>BS</th>
<th>Heston</th>
<th>BCC</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>$0.99</td>
<td>$0.79</td>
<td>$0.78</td>
<td>$0.52</td>
</tr>
<tr>
<td>Median</td>
<td>$0.98</td>
<td>$0.70</td>
<td>$0.70</td>
<td>$0.51</td>
</tr>
<tr>
<td>Std.Dev.</td>
<td>$0.42</td>
<td>$0.40</td>
<td>$0.40</td>
<td>$0.19</td>
</tr>
</tbody>
</table>

Panel B – Out of Sample

<table>
<thead>
<tr>
<th>RMSE</th>
<th>BS</th>
<th>Heston</th>
<th>BCC</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>$1.03</td>
<td>$0.88</td>
<td>$0.88</td>
<td>$0.62</td>
</tr>
<tr>
<td>Median</td>
<td>$1.02</td>
<td>$0.81</td>
<td>$0.81</td>
<td>$0.57</td>
</tr>
<tr>
<td>Std.Dev.</td>
<td>$0.44</td>
<td>$0.44</td>
<td>$0.44</td>
<td>$0.27</td>
</tr>
</tbody>
</table>

Panel C – Short Maturities

<table>
<thead>
<tr>
<th>RMSE</th>
<th>BS</th>
<th>Heston</th>
<th>BCC</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>$0.62</td>
<td>$0.29</td>
<td>$0.28</td>
<td>$0.14</td>
</tr>
<tr>
<td>Median</td>
<td>$0.51</td>
<td>$0.18</td>
<td>$0.18</td>
<td>$0.11</td>
</tr>
<tr>
<td>Std.Dev.</td>
<td>$0.37</td>
<td>$0.29</td>
<td>$0.29</td>
<td>$0.12</td>
</tr>
</tbody>
</table>

Panel D – Medium Maturities

<table>
<thead>
<tr>
<th>RMSE</th>
<th>BS</th>
<th>Heston</th>
<th>BCC</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>$1.31</td>
<td>$0.87</td>
<td>$0.87</td>
<td>$0.30</td>
</tr>
<tr>
<td>Median</td>
<td>$1.07</td>
<td>$0.63</td>
<td>$0.63</td>
<td>$0.22</td>
</tr>
<tr>
<td>Std.Dev.</td>
<td>$0.90</td>
<td>$0.81</td>
<td>$0.81</td>
<td>$0.23</td>
</tr>
</tbody>
</table>

Panel E – Long Maturities

<table>
<thead>
<tr>
<th>RMSE</th>
<th>BS</th>
<th>Heston</th>
<th>BCC</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>$1.82</td>
<td>$1.25</td>
<td>$1.24</td>
<td>$0.46</td>
</tr>
<tr>
<td>Median</td>
<td>$1.50</td>
<td>$0.91</td>
<td>$0.91</td>
<td>$0.39</td>
</tr>
<tr>
<td>Std.Dev.</td>
<td>$1.22</td>
<td>$1.12</td>
<td>$1.12</td>
<td>$0.32</td>
</tr>
</tbody>
</table>
### Table 4
RMSE (Root Mean Square Errors) Comparison between Various Models

For each pair of models, the table presents the percentage of days when one model outperforms (has a smaller RMSE) and underperforms (has a larger RMSE) another model. It also presents the difference in average RMSEs in each case. Panel A uses in-sample tests where the daily RMSEs are calculated from the prediction errors for the prices of the option contracts under the parameter set that minimizes the RMSE for the same day. Panel B reports the same statistics when the parameters for each day are estimated from contracts on the previous day. Panel C reports the corresponding statistics for the three maturity sub-samples [short (7 ~ 39 days), medium (42 ~ 130), and long (133 ~ 221 days)] for the Black-Scholes model and for our model where the RMSEs are re-estimated for each sub-sample.

#### Panel A – In Sample

<table>
<thead>
<tr>
<th>outperforming model</th>
<th>bs</th>
<th>heston</th>
<th>bcc</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>bs</td>
<td></td>
<td>100%</td>
<td>100%</td>
<td>84%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.42</td>
<td>$0.42</td>
<td>$1.13</td>
</tr>
<tr>
<td>heston</td>
<td>0%</td>
<td></td>
<td>100%</td>
<td>72%</td>
</tr>
<tr>
<td></td>
<td>$0.00</td>
<td></td>
<td>$0.06</td>
<td>$0.79</td>
</tr>
<tr>
<td>bcc</td>
<td>0%</td>
<td>0%</td>
<td></td>
<td>72%</td>
</tr>
<tr>
<td></td>
<td>$0.00</td>
<td>$0.00</td>
<td></td>
<td>$0.79</td>
</tr>
<tr>
<td>our</td>
<td>16%</td>
<td>28%</td>
<td>28%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.23</td>
<td>$0.28</td>
<td>$0.28</td>
<td></td>
</tr>
</tbody>
</table>

#### Panel B – Out of Sample

<table>
<thead>
<tr>
<th>outperforming model</th>
<th>bs</th>
<th>heston</th>
<th>bcc</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>bs</td>
<td></td>
<td>89%</td>
<td>89%</td>
<td>83%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.39</td>
<td>$0.39</td>
<td>$1.11</td>
</tr>
<tr>
<td>heston</td>
<td>11%</td>
<td></td>
<td>100%</td>
<td>73%</td>
</tr>
<tr>
<td></td>
<td>$0.00</td>
<td></td>
<td>$0.05</td>
<td>$0.86</td>
</tr>
<tr>
<td>bcc</td>
<td>11%</td>
<td>0%</td>
<td></td>
<td>73%</td>
</tr>
<tr>
<td></td>
<td>$0.00</td>
<td>$0.00</td>
<td></td>
<td>$0.86</td>
</tr>
<tr>
<td>our</td>
<td>17%</td>
<td>27%</td>
<td>27%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.24</td>
<td>$0.26</td>
<td>$0.27</td>
<td></td>
</tr>
</tbody>
</table>
Panel C – Short Maturities

<table>
<thead>
<tr>
<th></th>
<th>outperforming model</th>
<th>bs</th>
<th>heston</th>
<th>bcc</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>bs</td>
<td>100%</td>
<td>$0.35</td>
<td>$0.35</td>
<td>$0.46</td>
<td></td>
</tr>
<tr>
<td>heston</td>
<td>0%</td>
<td>$0.00</td>
<td>100%</td>
<td>68%</td>
<td></td>
</tr>
<tr>
<td>bcc</td>
<td>0%</td>
<td>$0.00</td>
<td>0%</td>
<td>67%</td>
<td></td>
</tr>
<tr>
<td>our</td>
<td>3%</td>
<td>$0.07</td>
<td>32%</td>
<td>33%</td>
<td></td>
</tr>
</tbody>
</table>

Panel D – Medium Maturities

<table>
<thead>
<tr>
<th></th>
<th>outperforming model</th>
<th>bs</th>
<th>heston</th>
<th>bcc</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>bs</td>
<td>100%</td>
<td>$0.49</td>
<td>$0.49</td>
<td>$1.18</td>
<td></td>
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<tr>
<td>heston</td>
<td>0%</td>
<td>$0.00</td>
<td>100%</td>
<td>73%</td>
<td></td>
</tr>
<tr>
<td>bcc</td>
<td>0%</td>
<td>$0.00</td>
<td>0%</td>
<td>73%</td>
<td></td>
</tr>
<tr>
<td>our</td>
<td>8%</td>
<td>$0.23</td>
<td>27%</td>
<td>27%</td>
<td></td>
</tr>
</tbody>
</table>

Panel E – Long Maturities

<table>
<thead>
<tr>
<th></th>
<th>outperforming model</th>
<th>bs</th>
<th>heston</th>
<th>bcc</th>
<th>our</th>
</tr>
</thead>
<tbody>
<tr>
<td>bs</td>
<td>100%</td>
<td>$0.67</td>
<td>$0.67</td>
<td>$1.80</td>
<td></td>
</tr>
<tr>
<td>heston</td>
<td>0%</td>
<td>$0.00</td>
<td>100%</td>
<td>67%</td>
<td></td>
</tr>
<tr>
<td>bcc</td>
<td>0%</td>
<td>$0.00</td>
<td>0%</td>
<td>66%</td>
<td></td>
</tr>
<tr>
<td>our</td>
<td>18%</td>
<td>$0.30</td>
<td>33%</td>
<td>34%</td>
<td></td>
</tr>
</tbody>
</table>
Table 5  Regression Results of Profits from Selling Naked Call Options:

<table>
<thead>
<tr>
<th>Equations</th>
<th>Profit Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a.</strong></td>
<td>( P = a + b_1 s' + b_2 r + b_3 \ln(T - t) + e ),</td>
</tr>
<tr>
<td><strong>b.</strong></td>
<td>( P = a + b_2 \frac{\hat{v}}{\sqrt{T - t}} + b_3 r + b_4 \ln(T - t) + e ),</td>
</tr>
<tr>
<td><strong>c.</strong></td>
<td>( P = a + b_3 \frac{\hat{V}_{	ext{Hes}}}{T - t} + b_4 r + b_5 \ln(T - t) + e ),</td>
</tr>
<tr>
<td><strong>d.</strong></td>
<td>( P = a + b_4 \frac{\hat{V}_{	ext{BCC}}}{T - t} + b_5 r + b_6 \ln(T - t) + e ),</td>
</tr>
<tr>
<td><strong>e.</strong></td>
<td>( P = a + b_5 s' + b_6 \frac{\hat{v}}{\sqrt{T - t}} + b_7 r + b_8 \ln(T - t) + e ),</td>
</tr>
<tr>
<td><strong>f.</strong></td>
<td>( P = a + b_6 s' + b_7 \frac{\hat{V}<em>{	ext{Hes}}}{T - t} + b_8 \frac{\hat{V}</em>{	ext{BCC}}}{T - t} + b_9 r + b_{10} \ln(T - t) + e ),</td>
</tr>
<tr>
<td><strong>g.</strong></td>
<td>( P = a + b_9 r + b_{10} \ln(T - t) + b_{11} D + e ).</td>
</tr>
</tbody>
</table>

where \( P \) is the realized profit, \( \hat{v} / \sqrt{T - t} \) is the annualised implied volatility using our model, \( s' \) is the implied volatility using the Black-Scholes model, \( \hat{V}_{\text{Hes}} \) is the variance of the Heston (1993) model, \( \hat{V}_{\text{BCC}} \) is the variance of the Bakshi-Cao-Chen (1997) model, \( r \) is the realized index returns, and \( D \) is the dollar difference that corresponds to the difference between the volatilities implied by the two models. The t statistic is reported below each corresponding coefficient.
<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>106.17</td>
<td>126.32</td>
<td>113.11</td>
<td>113.03</td>
<td>113.71</td>
<td>109.57</td>
<td>120.88</td>
</tr>
<tr>
<td>b1</td>
<td>-7.30</td>
<td></td>
<td></td>
<td></td>
<td>-21.92</td>
<td>-21.92</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-13.08</td>
<td></td>
<td></td>
<td></td>
<td>-33.38</td>
<td>-33.38</td>
<td></td>
</tr>
<tr>
<td>b2</td>
<td>5.17</td>
<td>25.31</td>
<td></td>
<td></td>
<td>9.68</td>
<td>9.64</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>39.97</td>
<td>39.76</td>
<td></td>
</tr>
<tr>
<td>b3</td>
<td></td>
<td>-0.01</td>
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<td></td>
<td>-2.35</td>
<td></td>
<td></td>
</tr>
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<td>-0.48</td>
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<td></td>
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</tr>
<tr>
<td>b4</td>
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<td></td>
<td>-0.01</td>
<td></td>
<td>2.33</td>
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<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-0.42</td>
<td></td>
<td></td>
<td>4.37</td>
<td></td>
</tr>
<tr>
<td>b5</td>
<td>-215.73</td>
<td>-219.16</td>
<td>-216.68</td>
<td>-216.70</td>
<td>-218.32</td>
<td>-218.23</td>
<td>-217.00</td>
</tr>
<tr>
<td>b6</td>
<td>4.10</td>
<td>6.16</td>
<td>4.32</td>
<td>4.32</td>
<td>7.14</td>
<td>7.13</td>
<td>4.25</td>
</tr>
<tr>
<td></td>
<td>76.35</td>
<td>69.21</td>
<td>82.88</td>
<td>82.82</td>
<td>77.34</td>
<td>76.37</td>
<td>83.25</td>
</tr>
<tr>
<td>b7</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.03</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>23.45</td>
</tr>
<tr>
<td>R2 (adj)</td>
<td>0.76</td>
<td>0.76</td>
<td>0.76</td>
<td>0.76</td>
<td>0.77</td>
<td>0.77</td>
<td>0.76</td>
</tr>
<tr>
<td># of obs.</td>
<td>32,517</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 6  Regression Results of Profits from Selling Naked Call Options:

\[
eq 27 \quad \begin{align*}
a. P &= a + b s' \sqrt{T - t} + b_2 r + b_3 \ln[T - t] + e, \\
b. P &= a + b_2 v + b_3 r + b_4 \ln[T - t] + e, \\
c. P &= a + b_2 v'_T + b_3 r + b_4 \ln[T - t] + e, \\
d. P &= a + b_4 r + b_5 \ln[T - t] + b_6 D + e.
\end{align*}
\]

where \(P\) is the realized profit of the short call, \(\sqrt{T - t}\) is the annualised implied volatility using our model, \(r\) is the realized index returns, \(s'\) is the implied volatility using the Black-Scholes model, and \(D\) is the dollar difference that corresponds to the difference between the volatilities implied by the two models. The t statistic is reported to the right of each corresponding coefficient.

<table>
<thead>
<tr>
<th>Short Maturity (# of obs. 10,788)</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>intercept</td>
<td>-0.59</td>
<td>-1.92</td>
<td>-4.62</td>
<td>-9.29</td>
</tr>
<tr>
<td>b1</td>
<td>-3.80</td>
<td>-9.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b2</td>
<td></td>
<td>0.91</td>
<td>5.91</td>
<td>2.84</td>
</tr>
<tr>
<td>b5</td>
<td>-232.68</td>
<td>-186.22</td>
<td>-233.28</td>
<td>-186.10</td>
</tr>
<tr>
<td>b6</td>
<td>1.15</td>
<td>13.09</td>
<td>1.91</td>
<td>15.15</td>
</tr>
<tr>
<td>b7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R2 (adj)</td>
<td>0.7647</td>
<td>0.7636</td>
<td>0.7694</td>
<td>0.7642</td>
</tr>
</tbody>
</table>
### Medium Maturity (# of obs. 16,749)

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>intercept</td>
<td>-15.00</td>
<td>-19.24</td>
<td>-31.84</td>
<td>-34.41</td>
</tr>
<tr>
<td>b1</td>
<td>-22.49</td>
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R2 (adj) | 0.7634 | 0.7605 | 0.7705 | 0.7644 |

### Long Maturity (# of obs. 4,980)

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R2 (adj) | 0.8057 | 0.8045 | 0.8220 | 0.8096 |
Figure 1a

Implied Volatility Plots of the Black-Scholes (1973) Model and Our Model for Short Dated Options

Figure 1b

Implied Volatility Plots of the Black-Scholes (1973) Model and Our Model for Medium Dated Options
Figure 1c

Implied Volatility Plots of the Black-Scholes (1973) Model and Our Model for Long Dated Options