Pricing the Term Structure of Inflation Risk Premia: Theory and Evidence from TIPS*

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Abstract

In this paper, we study inflation risk and the term structure of inflation risk premia in U.S. nominal interest rates through the Treasury Inflation Protection Securities (TIPS) with a multi-factor, modified quadratic term structure model with correlated real and inflation rates. We derive closed form solutions to the real and nominal term structures of interest rates that drastically facilitate the estimation of model parameters and improve the accuracy of the valuation of nominal rates and TIPS. In addition, we contribute to the literature by estimating the term structure of inflation risk premia implied by TIPS. The empirical evidence using data from the period of January 1998 through October 2007 indicates that the expected inflation rate, contrary to those derived from the consumer price indexes, is very stable and the inflation risk premia exhibit a positive term structure.

Keywords: Quadratic term structure model of interest rates, TIPS, Unscented Kalman filter, Inflation risk premium

JEL Classification: C51, E31, E43, G12
Pricing the Term Structure of Inflation Risk Premia: Theory and Evidence from TIPS

I. Introduction

Inflation always plays an essential role in our economy. Having experienced the high inflation era due to the oil crisis in the 70’s, many countries have since started introducing inflation-protected securities (see Table 1). The fear of inflation has put hedging inflation risk a top priority in the investment world. In Israel, the inflation-indexed bonds account for over 80% of the total fixed income market!

Inflation poses a challenge for all investors, especially the holders of fixed income investments. For investors who rely on the stability and predictability of fixed income investing, finding ways to limit or mitigate the effects of inflation is crucial for continued financial security. Because of the surge of investors’ demand, inflation-indexed security market has been growing quickly.¹ Inflation-linked securities have been widely accepted by many institutional and individual investors.² Treasury Inflation Protection Securities (TIPS) in the United States were first introduced in January 1997 and expanded quickly. As of October 2007, a total of 25 TIPS have been traded (including two expired TIPS) in the market, and the TIPS market has grown to over $310 billion, or 6.3% of the total outstanding treasury debt by the end of 2007.

Although TIPS were only introduced to the U.S. in 1997, the inflation-indexed bonds have a long history. The world’s first known inflation-indexed bonds were issued by the Commonwealth of Massachusetts in 1780 during the Revolutionary War, when those bonds were invented to deal with severe wartime inflation and with angry discount among soldiers in the U.S. army with the

¹ Inflation protected securities are also issued by private entities. FHLMC (Freddie Mac), Tennessee Valley Authority Power, Household Finance, and John Hancock Life Insurance Company issue corporate bonds that are also indexed to CPI-U (unlike TIPS where the principal amount adjusted; these inflation protected corporate notes feature an adjustment of coupon rate). CPI indexed annuities are also available from Irish Life Company of North America (ILONA) and TIAA-CREF. Inflation protected CD that is FDIC insured are also available.
² For example, pension funds are stampeding into the inflation-linked market as a natural match for their long-dated liabilities; life insurance companies want a hedge against policies linked to inflation; and investors are seeking portfolio diversification
decline in purchasing power of their pay. While the appearance of the first inflation-indexed bond dated back in 1780, inflation-indexed bonds did not flourish until the twentieth century.

In the U.S., the principal amount of TIPS is adjusted on a daily basis for changes in the level of inflation; however, the inflation adjustment is not payable until maturity. The index used for determining the adjustment is the three-month lagged non-seasonally adjusted Consumer Price Index – Urban Consumer (CPI-U), published monthly by the U.S. Department of Labor. Since the interest payments of the TIPS are “inflation-protected,” the difference between yields of the TIPS and yields of the corresponding nominal bonds is usually perceived as the expected inflation. Indeed, the financial industry calculates the expected inflation by simply bootstrapping out the difference of the two. However, this crude approximation ignores the inflation risk premium and the fact inflation and real interest rates are highly correlated. Figure 1 depicts the 72-month running correlation between the inflation rate of CPI-U and the 1-year CMT rate. The correlation during the oil crisis in the 70’s reached 70%.

In this paper, we derive and test a two-factor term structure model for the inflation and the real rate. We assume that the first factor explains the prices of TIPS completely and the sum of the two factors explains the nominal interest rates that are proxied by CMTs (Constant Maturity Treasuries). Theoretically, our model is similar to Richard’s model (1978) where the nominal rate is approximated by the Fisher equation. However, our model differs from the Richard model in that our “inflation factor” is not inflation itself. Rather, it is a factor that mimics inflation closely. Formally, our inflation factor explains the difference between nominal and real rates. Our model is different from the Richard model also for the incorporation of the correlation between the factors. Technically, our model can be regarded as a special case of the Duffie-Kan (1996) which is in turn a special case of the Duffie-Pan-Singleton model (2000). The Duffie-Pan-Singleton model solves a rich class of payoff functions for fixed income derivatives that include the payoffs of default-free coupon and zero coupon bonds. By concentrating on this special case of the Duffie-Pan-Singleton model, we are able to provide true closed form solutions as opposed to those solutions that require solving a series of ODEs (Ordinary Differential Equations).

Empirically, our model directly compares to Jarrow and Yildirim (2003) who price TIPS and ordinary Treasuries with the HJM model. However, due to the limitation of the HJM model, they can only perform cross-sectional fitting and hence cannot estimate inflation risk premium. More importantly, they do not estimate the correlation endogenously, which is inconsistent with

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3 Interested readers may refer to Shiller (2003) for more historical information of inflation-indexed bonds.
4 The exact details of the protection will be discussed later.
the rest of the estimation methodology. In contrast, our model can be estimated endogenously both the correlation between the real rate and inflation and the inflation risk premium.

The paper is organized as follows. A brief literature review of recent term structure model and inflation is given in Section II. In Section III, we derive a true closed form solution that requires no ODE solution to the modified quadratic term structure model where the two factors – real interest rate and inflation rate are correlated. In Section IV, the technique used to estimate our model with TIPS and nominal CMT rates is discussed. The empirical results are described in Section V and finally Section VI gives the conclusion.

II Related Literature

This paper is motivated by two strands of the literature in inflation. One is the literature on the nominal term structure where the nominal rate is approximated by the real rate and inflation, known as the Fisher approximation. This is a “reduced form” approach to model inflation. The other literature takes the “structural” approach where inflation is endogenously determined in a monetary economy. Using the continuous time methodology for the first time, CIR (1985) unite the two approaches and derive a multi-factor general equilibrium model for the nominal bond in which an exact relationship between real rate, inflation, and nominal rate is defined and the Fisher equation is validated as the first order approximation.

Following CIR, models for the nominal term structure have grown in three directions. The first set of models assumes multiple factors that explain the nominal interest rates but do not explicitly assume the factors to be real rate and inflation but rather leave the factors unspecified. These include, among many others, (i) Langetieg (1980), Hull and White (1990), Turnbull and Milne (1991), Longstaff and Schwartz (1992), Duffie and Kan (1996), Dai and Singleton (2000), and Duffie, Pan, and Singleton (2000) under the affine formulation, (ii) Inui and Kijima (1998) under the HJM formulation, and (iii) Ahn, Dittmar, and Gallant (2002), Leippold and Wu (2002), and Kim (2004) under the quadratic formulation. Our model can be regarded as a special case of the Duffie-Kan, Dai-Singleton or Duffie-Pan-Singleton model, which is less general. However, it grants us a true closed form solution (as opposed to a series of ordinary differential equations) which provides improved efficiency and accuracy in the empirical study.

The second approach adopts the Fisher approximation directly and decomposes the nominal rate into real rate, inflation, and sometimes inflation risk premium. These include Richard (1978), Jarrow and Yildirim (2003) and Ang and Bekaert (2004). These models all assume independence

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5 Note that the instantaneous nominal rate is the sum of the instantaneous real rate and the instantaneous inflation rate. There is no inflation risk premium. Risk premia exist for term rates.
between the real rate and inflation in order to gain mathematical tractability. However, unlike the previous approach where the factors are left unspecified, it is very hard to justify the assumption of the independence. Our model solves this problem because it explicitly incorporates the correlation between the real rate and inflation in the model while the closed form solution still exists.

The third approach turns to the fundamental economy of the CIR model and derives models for inflation. These include Bakshi and Chen (1996) and Buraschi and Jiltsov (2005). Bakshi and Chen (1996) develop a general equilibrium model in the framework of the CIR economy in which production output and money supply are the only two underlying state variables. In this framework, they find the nominal and real term structures to present completely different properties, including the fundamental risk structure. They provide no empirical study. Buraschi and Jiltsov (2004) focus on the inflation risk premia. They derive a normal bond price solution in the framework of monetary supply and real productivity. However, their empirical analysis is based on the information of non-market information such as an inflation index and money supply.

Empirical work on the nominal term structure and inflation is very diverse and hard to synthesize. First of all, many models do not specify what the factors are. For example, see Chen and Scott (1993, 2003), Babbs and Nowman (1999), and Dai and Singleton (2000). Also, many empirical studies in estimating inflation and its risk premium use survey or inflation index (e.g. CPI) data which are detached totally (for example, see Evans (1998, 2003)) or partially (see Sun (1992), Gibbons and Ramaswamy (1993), and Brennan, Wang, and Xia (2004)) from the term structure theory and hence difficult to estimate the inflation risk premium. Some studies use inflation-indexed and nominal bonds but then they lack the equilibrium model to provide a theoretically structural justification. For example, see Brown and Shafer (1994), Campbell and Shiller (1996) and Barr and Campbell (1997).

So far, to our best knowledge, there has been no study that estimates the parameters of the nominal term structure model jointly from the market prices. The closest study is Jarrow and Yildirim (2003) but due to an identification problem in the HJM model the correlation parameter is estimated by historical time series. Our model solves the correlation problem in Richard (1978) and Ang and Bekaert (2004) and does not suffer the identification problem in HJM-like models, and hence all parameters can be estimated by market prices of real (TIPS) and nominal (Treasuries) bond prices. In particular, time series and cross-sectional data allow us to estimate the correlation and inflation risk premium. Most importantly, having a direct measure of the spread between nominal and real bond yields can vastly improve the precision of the estimations of the inflation risk premium.
III The Model

In their last section, Cox, Ingersoll, and Ross (1985) lay out a theory of real rate and inflation and provide a solution for the nominal bond. In their equation 60, they derive an equilibrium result where the (instantaneous) nominal rate is decomposed into the (instantaneous) real rate and an inflation factor. The inflation factor composes of a set of components of the random price factor and the state variables. Here, we simplify and assume that the inflation factor itself follows a random process. Formally, we assume the nominal interest rate (instantaneous) \( R(t) \) is a sum of the real rate \( r(t) \) and an inflation factor \( i(t) \) as follows:

\[
(1) \quad R(t) = r(t) + i(t)
\]

where

\[
(2) \quad r(t) = \beta(\Theta, t) + \chi(t)
\]

\[
(3) \quad i(t) = b(\Theta, t) + k(t)
\]

where \( \beta(\Theta, t) \) and \( b(\Theta, t) \) are deterministic functions dependent upon time and parameters included in the parameter space \( \Theta \); \( \chi(t) = x(t), k(t) = y(t) \), and

\[
\begin{align*}
&dx(t) = \frac{1}{2}[-\alpha x(t)dt + \sigma dB_1(t)] \\
&dy(t) = \frac{1}{2}[-a y(t)dt + g\{ \rho dB_1(t) + \sqrt{1-\rho^2} dB_2(t) \}]
\end{align*}
\]

where \( \alpha, \sigma, a, g, \) and \( \rho \) are constants and \( dB_1 dB_2 = 0 \) under the risk neutral \( \mathbb{Q} \) measure.

By Ito’s lemma, we know that

\[
\begin{align*}
&d\chi(t) = (\frac{1}{2}\sigma^2 - \alpha \chi(t))dt + \sigma \sqrt{\chi(t)} dB_1(t) \\
&dk(t) = (\frac{1}{2}g^2 - a k(t))dt + g \sqrt{k(t)} \{ \rho dB_1(t) + \sqrt{1-\rho^2} dB_2(t) \}
\end{align*}
\]

Each equation in (4) is used for the quadratic term structure modeling and is also a special case of the square root process used by CIR. Therefore, the real bond price (subscript \( r \)) can be expressed as follows:

\[
(5) \quad P_r(0, T) = E_0 \left[ \exp \left( -\int_0^T r(u)du \right) \right] = \exp \left( -\int_0^T \beta(\Theta, u)du \right) E_0 \left[ \exp \left( -\int_0^T \chi(u)du \right) \right] = \exp \left( -\int_0^T \beta(\Theta, u)du \right) \Phi(0, T) e^{-\lambda(t)/\theta(0, T)}
\]

where

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\( ^6 \) Section 7: Uncertain Inflation and the Pricing of Nominal Bonds, page 401.

\( ^7 \) The parameters used in the empirical study are mean reversion speed \( <\alpha, a> \), reversion level \( <\mu, m> \), volatility \( <\sigma, g> \), market price of risk \( <\lambda, q> \), and correlation \( \rho \).
\[ \Phi(0,T) = \left[ \frac{2\xi e^{(\alpha + \xi)T/2}}{(\alpha + \xi)(e^{\xi T} - 1) + 2\xi} \right]^{\frac{1}{2}} \]

\[ G(0,T) = \frac{2(e^{\xi T} - 1)}{(\alpha + \xi)(e^{\xi T} - 1) + 2\xi} \]

\[ \xi = \sqrt{\alpha^2 + 2\sigma^2} \]

In order to replicate the CIR closed form solution, we propose the following deterministic terms in each factor:

\[ \beta(\Theta, t) = \left( \frac{1}{2} - \frac{2\alpha \mu}{\sigma^2} \right) \frac{\alpha + \xi}{2} - \frac{(\alpha + \xi) \xi e^{\Theta t}}{(\alpha + \xi)(e^{\xi T} - 1) + 2\xi} \]

\[ r(0) = \chi(0) \quad (\beta(\Theta, 0) = 0) \]

where \( \mu \) is a constant representing the reverting level in the CIR model. As a result,

\[ \exp \left( -\int_0^T \beta(\Theta, u) du \right) = \left[ \frac{2\xi e^{(\alpha + \xi)\frac{T}{2}}}{(\alpha + \xi)(e^{\xi T} - 1) + 2\xi} \right]^{\frac{1}{\alpha + \xi}} \]

Equation (5) hence becomes:

\[ P_t(0, T) = F(0, T)e^{-r(t)G(1, T)} \]

where

\[ F(0, T) = \left[ \frac{2\xi e^{(\alpha + \xi)\frac{T}{2}}}{(\alpha + \xi)(e^{\xi T} - 1) + 2\xi} \right]^{\frac{1}{\alpha + \xi}} \]

The nominal (subscript \( n \)) risk-free, zero-coupon bond with maturity of \( T \) is:

\[ P_n(0, T) = E_0 \left[ \exp \left( -\int_0^T R(u) du \right) \right] \]

\[ = E_0 \left[ \exp \left( -\int_0^T [r(u) + \delta(u)] du \right) \right] \]

\[ = E_0 \left[ \exp \left( -\int_0^T [\beta(\Theta, u) + b(\Theta, u)] du \right) \exp \left( -\int_0^T [x^2(u) + y^2(u)] du \right) \right] \]

Given that \( x(t) \) and \( y(t) \) are correlated, the expectation of (9) cannot be computed easily. More general versions of the above problem have been solved in the literature. Nevertheless, those solutions rely on solving a system of ordinary differential equations. As we shall show shortly, (9) has a closed form solution.

For the purpose of exposition (and extension) we define the following terms as:
\[ X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1/2 \sigma & 0 \\ 1/2 \varrho & 1/2 \sqrt{1-\rho^2} \end{bmatrix} \]

Then the SDE of the interest rates in (3) can be rewritten as:

\[(10) \quad dX(t) = AX(t)dt + \Sigma dB(t)\]

To solve (9), we define a new measure \( \tilde{Q} \) so that:

\[(11) \quad \int_0^T [\beta(\Theta, u) + b(\Theta, u)]du \mathcal{E}_0 \int_0^T \exp \left[ -\int_0^T [x^2(u) + y^2(u)]du \right] \]

where Radon-Nykodim derivative is defined as (\( \Gamma \) is to be determined):

\[(12) \quad \frac{d\tilde{Q}}{dQ} = \exp \left[ -\int_0^T X'(u) \Gamma(u) \Sigma dB(u) - \frac{1}{2} \int_0^T X'(u) \Gamma(u) \Sigma \Sigma' \Gamma'(u) X(u)du \right] \]

Using Ito’s lemma on the first term of the last line of (12), we get:

\[(13) \quad d[X'(u) \Gamma(u) X(u)] = X'(u) \Gamma(u) dX(u) + [X'(u)] \Gamma(u) X(u) + X'(u) \frac{\partial \Gamma(u)}{\partial u} X(u)du \]

which gives:

\[(14) \quad X'(u) \Gamma(u) dX(u) = \frac{1}{2} d[X'(u) \Gamma(u) X(u)] - \frac{1}{2} X'(u) \frac{\partial \Gamma(u)}{\partial u} X(u)du - \frac{1}{2} \text{tr}[\Sigma' \Gamma(u) \Sigma]du \]

Substituting (14) back into (12), we get:

\[(15) \quad \frac{d\tilde{Q}}{dQ} = \exp \left[ -\int_0^T X'(T) \Gamma(T) X(T) + \frac{1}{2} X'(0) \Gamma(0) X(0) + \int_0^T \frac{1}{2} X'(u) \frac{\partial \Gamma(u)}{\partial u} X(u)du \right. \]

Substituting (15) back into (11), we get:
\[ P_u(0,T) = \exp\left(-\int_0^T [\beta(\Theta,u) + b(\Theta,u)]du\right) E_0 \left[ \exp\left(-\int_0^T (x(u)^2 + y(u)^2)du\right) \right] \]
\[ = \exp\left(-\int_0^T [\beta(\Theta,u) + b(\Theta,u)]du\right) E_0 \left[ \frac{d\tilde{Q}}{d\tilde{Q}} \exp\left(-\int_0^T (x(u)^2 + y(u)^2)du\right) \right] \]
\[ = \exp\left(-\int_0^T [\beta(\Theta,u) + b(\Theta,u)]du\right) E_0 \left[ \exp\left[\frac{1}{2} X'(T)\Gamma(T)X(T) - \frac{1}{2} X'(0)\Gamma(0)X(0) \right] \right. \]
\[ - \int_0^T \frac{1}{2} X'(u) \left[ \frac{\partial \Gamma(u)}{\partial u} + \Gamma(u)A + A\Gamma'(u) - \Gamma(u)\Sigma\Sigma'\Gamma'(u) + 2I \right] X(u)du \]
\[ - \left. \int_0^T \frac{1}{2} tr[\Sigma'\Gamma(u)\Sigma] du \right] \]
\[(16)\]

Note that the \( \Gamma(t) \) function is arbitrary, which is the free degree of freedom we have in the model. Hence, we can set it so that the quadratic term of the second line of the final expression of (16) is equal to 0. Hence, we arrive at the following ODE:
\[ (17) \quad \frac{\partial \Gamma(u)}{\partial u} + \Gamma(u)A + A\Gamma'(u) - \Gamma(u)\Sigma\Sigma'\Gamma'(u) + 2I = 0 \]
where \( I \) is a 2x2 identity matrix. Under the condition that \( A \) is symmetric so it is convenient that we can also assume \( \Gamma \) to be symmetric. Then, (17) is equal to:
\[ (17') \quad \frac{\partial \Gamma(u)}{\partial u} + \Gamma(u)A + A\Gamma'(u) - \Gamma(u)\Sigma\Sigma'\Gamma'(u) + 2I = 0 \]
Equation (17') is known as the Riccati equation with the terminal condition \( \Gamma(T) = 0 \). Define \( N(u - T) \) to be an exponential matrix used in the solution of Riccati equation.
\[ (18) \quad N(u - T) = \begin{bmatrix} n_1(u - T) & n_2(u - T) \\ n_3(u - T) & n_4(u - T) \end{bmatrix} = \exp \begin{bmatrix} -A & -2I \\ -\Sigma\Sigma^T & A \end{bmatrix} \times (u - T) \]
Then the solution of (17') is given by:
\[ (19) \quad \Gamma(u) = n_2(u - T)n_4^{-1}(u - T) \]
where
\[ n_2(u - T) = \begin{bmatrix} I_{2 \times 2} & 0_{2 \times 2} \end{bmatrix} N(u - T) \begin{bmatrix} 0_{2 \times 2} \\ I_{2 \times 2} \end{bmatrix} \]
\[ n_4(u - T) = \begin{bmatrix} 0_{2 \times 2} & I_{2 \times 2} \end{bmatrix} N(u - T) \begin{bmatrix} 0_{2 \times 2} \\ I_{2 \times 2} \end{bmatrix} \]
Using the results of (17) and (19), (16) can be simplified to the following equation,
\[ (20) \]

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See Levin (1959).
\[ P_u(0, T) = \exp\left( -\int_0^T [\beta(\Theta, u) + b(\Theta, u)]du \right) \mathbb{E}_0 \left[ \frac{dQ}{dQ} \exp\left( -\int_0^T (x(u)^2 + y(u)^2)du \right) \right] \]

\[ = \exp\left( -\int_0^T [\beta(\Theta, u) + b(\Theta, u)]du \right) \exp\left( -\frac{1}{2} X'(0) \Gamma(0) X(0) - \int_0^T \frac{1}{2} \text{tr} \left[ \Sigma'(u) \Sigma(0) \right] \right) \]

\[ = \left[ \frac{2\xi e^{(\alpha + \xi)T/2}}{(\alpha + \xi)(e^{\xi T} - 1) + 2\xi} \right]^{\frac{2\alpha - 1}{\tau - 1}} \left[ \frac{2c e^{(\alpha + c)T/2}}{(\alpha + c)(e^{c T} - 1) + 2c} \right]^{\frac{2\alpha - 1}{\tau - 1}} \exp\left( -\frac{1}{2} X'(0) \Gamma(0) X(0) - \int_0^T \frac{1}{2} \text{tr} \left[ \Sigma'(u) \Sigma(0) \right] \right) \]

where \( \Xi \) is a closed form solution given in the Appendix and \( c = \sqrt{a^2 + 2g^2} \). This is the closed form solution in the Black-Scholes sense. It is much more efficient to implement than solving a series of ODEs, as in Duffie and Kan (1996) and Duffie, Pan, and Singleton (2000). In the following section, we develop a two-stage estimation framework to estimate the parameters in the model. Firstly, we estimate the parameters of the real interest rate with TIPS. Secondly, with the closed-form solution of the nominal bond price described by equation (20), we can estimate the implied inflation risk parameters from nominal interest rates.

**IV Estimation Procedure**

In estimation of the term structure, past studies have used a variety of different methods. Cross-sectional estimation was used by Brown and Dybvig (1986), Brown and Schaefer (1994) and others. This method is known to have the limitation that the model is under-identified and not able to estimate the market price of risk or the risk premium. To resolve this problem, Chen and Scott (1993) and Pearson and Sun (1994) use cross sectional and time series data from Treasuries to estimate all the parameters including the risk premium. Furthermore, Chen and Scott (1993) and Pearson and Sun (1994) make use of the closed form solution of the bond price in the CIR model and the closed form solution of the state variable so that exact maximum likelihood estimation is possible. Unfortunately, one problem remains that the measurement errors must be restrictive. If all bonds are priced with flexibly specified measurement errors, then filtering methods must be used to obtain fitted states. The application of Kalman filter in the estimation of term structure models has been investigated by Pennacchi (1991), Chen and Scott (2003), Jegadeesh and Pennacchi (1996), Babbs and Nowman (1999), and Brennan, Wang and Xia (2004). The method of maximum likelihood estimation with Kalman filter permits the underlying state variables be handled correctly as unobservable variables.

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9 For details, see Chen and Scott (1993).
While the maximum likelihood estimation is efficient, it requires that the model must have a closed form solution to the discount bond price. Then, the implied state variables are obtained by inverting the model for these bond yields. However, such inversion is very difficult for some models such as the quadratic term structure model. For those models, Quasi-MLE (e.g. Chan, Karolyi, Longstaff, and Sanders (1992)) or moments-based estimation (e.g. Ahn, Dittmar, and Gallant (2002)) must be used instead. As Dai and Singleton (2000) point out, direct likelihood-based comparisons, even across different classes of \(N\)-factor affine models, are impossible, since the models are not nested.\(^{10}\)

Our model is part of the affine family and has a closed form solution. Hence, we can use the maximum likelihood estimation with Kalman filter. However, the solution to the nominal bond price, unlike the solution to the real bond price, contains correlation and is not directly invertible to obtain the state variables. Hence, we approximate the inversion with Unscented Kalman filter (UKF). We estimate the dynamics of real interest rate and inflation factors in two consecutive steps. In each step, we cast the model into a state-space form, obtain efficient forecasts on the conditional mean and variance of observed prices of TIPS and nominal CMT rates, and build the likelihood function on the forecasting errors of the observed series, assuming the forecasting errors are normally distributed.

**Estimation of the Real Interest Rate**

TIPS provide us good securities to derive the real interest rate. Like nominal treasury bonds, real interest rates (real yield) can be directly derived from the cash flows and prices of TIPS. In calculating the real yield, inflation can be ignored because the transaction price and coupons are based on the accrued face value. Thus, the real yield is simply the internal rate of return that equates the current price plus accrued real interest to the discounted future real payments. Similarly, when we calculate the real forward rate and the real spot rate, the inflation factor (CPI-U) is ignored. The price of a TIPS is described as:

\[
P_{\text{TIPS}}^k(0,T_t) = \sum_{j=1}^{J} \left( \bar{C}^k_I P_n(0,T_t) + \bar{F}^k_I P_n(0,T_t) \right)
\]

\[(21)\]

\[
= \sum_{j=1}^{J} \bar{C}^k P(0,T_t) + \bar{F}^k P(0,T_t)
\]

\[
= \sum_{j=1}^{J} \bar{C}^k P(0,T_t) + \bar{F}^k P(0,T_t)
\]

\(^{10}\) Duffee and Stanton (2004) find that despite its attractive asymptotic properties, EMM performs worse than QML in a small sample, especially in the case where a large number of moments are needed in EMM.
where $T_k = T_j$ is the maturity of the $k$th TIPS, $I_j$ is time-$T_j$ CPI-U inflation index, $C_j$ and $F(T_j)$ are coupon payment of the $j$th period and the principal at the maturity; $C$ and $F$ are initial semi-annual coupon payment and the face value of TIPS respectively, and finally, $P^*_t(0,t)$ and $P^*_t(0,t)$ are real and nominal discount factors defined in (5) and (9) respectively. In (21), we assume that $1/P^*_t(0,T_j) = P^*_t(0,T_j)$. In other words, we assume that the nominal bond price, after adjusted for inflation, is equivalent to the real bond price.

To get the implied real forward rates, we assume piecewise linear forward rates and strip out zeros from TIPS by minimizing the sum of squared errors (SSE) between the market and the model prices. Following Jarrow and Yildirim (2003), we assume four-segment piecewise constant real forward rate curve (zero-3 years, 3 years-5 years, 5 years-10 years, 10 years above), and each real forward rate $f_r(t,s)$ curve is estimated by minimizing the SSE of the following:

\[
\begin{align*}
\min_{\{f^{(3)}, \ldots, f^{(20)}\}} \sum_{k=1}^{\infty} P^*_t(t) - \left\{ \sum_{j=3,5,10,20} C_j \exp \left( - \sum_{t\leq j} f_r(t) \tau_j \right) + F(T_k) \exp \left( - \sum_{j=3,5,10,20} f_r(t) \tau_j \right) \right\}^2
\end{align*}
\]

where $f_r^{(j)}$ represents the $j$-year real forward rate and $\tau_j$ is the length of the $j$-th period. After we get the estimation of real forward rate $f_r^{(j)}$ from the above optimization, we calculate the implied constant maturity real rates as follows:

\[
\begin{align*}
\zeta_3 &= f_r^{(3)} \\
\zeta_5 &= \frac{3 \times f_r^{(3)} + 2 \times f_r^{(5)}}{5} \\
\zeta_{10} &= \frac{3 \times f_r^{(3)} + 2 \times f_r^{(5)} + 5 \times f_r^{(10)}}{10} \\
\zeta_{20} &= \frac{3 \times f_r^{(3)} + 2 \times f_r^{(5)} + 5 \times f_r^{(10)} + 10 \times f_r^{(20)}}{20}
\end{align*}
\]

and the corresponding zero coupon bond price is $P^*_t(0,t) = \exp \left\{ - t \times \zeta_j \right\}$ where $t \in \{3,5,10,20\}$.

Under the assumption of the model, the real rate follows a square root process. Therefore, the bond price in real term is the standard CIR result of (8) with $\hat{\alpha} = \hat{\alpha} + \lambda$ and $\alpha \mu = \hat{\alpha} \hat{\mu}$ in which $\hat{\alpha}$, $\hat{\mu}$, and $\lambda$ represent reverting speed, level, and market price of risk under the real measure respectively. Equation (23) is known as the bootstrap method. The implied constant maturity real rates are the counterparts of the nominal CMTs. The difference of the two (spread) is usually perceived as the expected inflation. Even though the spread is not equal to the expected inflation, it is usually regarded as such by the industry and it should represent a close proxy, at least qualitatively. Theoretically, such spreads contain more than just expected inflation. It also contains inflation risk premium. In a following section, we will study the Fisher equation more carefully where the nominal rates with various maturities are compared to the Fisher equation.
which is the sum of the real rate, inflation, and inflation risk premium of the same maturity. The results are presented in Figure 2.

In the state-space form, we regard the one real interest rate spot rate as the unobservable state and specify the state propagation using an Euler approximation of statistical dynamics of the real interest rate factor embedded in equation (3):

\[ x_t = \phi x_{t-1} + \sqrt{Q}\varepsilon_t \]

where \( \varepsilon_t \) are i.i.d. standard normal innovations, \( \Phi = \exp(-\frac{1}{2}\alpha\Delta t) \) and \( Q = \Delta t \), \( \Delta t = 7/365 \) corresponds to weekly frequency of the data. We construct the measurement equations based on the 13 observed TIPS price, assuming additive, normally-distributed measurement errors:

\[ Y_r^k = Y_r^k + e_r^k \]

where \( Y_r^k \) represent the theoretical price of the \( k \)-th TIPS, \( Y_r^k \) represents the market observed TIPS price, \( \text{var}[e_r^k] = \sigma^2 \) for all \( k \), and \( \text{cov}[e_r^k,e_r^j] = 0 \) for \( k \neq j \).

A convenient approach to deal with measurement errors is to cast the TIPS prices in a state space augmented by measurement equations that relate the observed prices to the underlying state variables. When the state variables are Gaussian and the measurement equations are linear, the Kalman filter yields the efficient state updates in the least square sense. However, in our application, the state propagation equation in equation (3) is Gaussian linear, but the measurement equation in (25) is nonlinear in the state variables. Traditional literature uses an extended version of the Kalman filter (EKF) by approximating the nonlinearity via a Taylor expansion\(^{11}\). However, since the ultimate objective is to obtain the posterior distribution of the state variables given the observations, Julier and Uhlmann (1997) propose the Unscented Kalman Filter (UKF) to directly approximate the posterior density using a set of deterministically chosen sample points (sigma points). These sample points completely capture the true mean and covariance of the Gaussian state variables. This approach is computationally efficient because it avoids the calculation of derivatives for the linear approximation. The method also improves the accuracy because it reduces the convexity bias induced in the first-order approximation in the EKF. We will discuss the UKF in the next section.

From the unscented Kalman filter we obtain efficient forecasts on the conditional mean \( \bar{y}_r \) and conditional covariance matrix \( \bar{V} \) of the 13 TIPS prices, and build the likelihood function based on the conditional density of the pricing errors:

\(^{11}\) Examples of EKF in term structure model estimation include Chen and Scott (2003), Duffee and Stanton (2004); Lu and Wu (2005) use the UKF in their empirical estimation.
(26)  \[
\max \limits_{\Theta} L_t(\Theta; \{y_t\}) = \sum_{t=1}^{N} \left[ -\frac{1}{2} \log | \mathcal{V}_t | - \frac{1}{2} \left( (y_t - \mathcal{V}_t^{-1} y_t) \right) \right] \quad \Theta = \{\alpha, \mu, \sigma, \lambda\}
\]

where \( y_t \) is a vector of 13 TIPS in the dataset at time \( t \).

**Estimation of Nominal Term Structure of Interest Rates**

In this step, we take the estimated real interest rate factor dynamics in the first step as given, and estimate the dynamics of the, \( i(t) \), using the term structure of nominal interest rate. Under our model specification, the instantaneous inflation rate follows a CIR process too. The closed form solution of correlated real rate and inflation rate model given by equation (20) enable us to estimate the parameter as well as the inflation risk premium more easily with UKF. The estimation of the nominal rate process is performed on weekly observations. We first note that nominal CMT rates are not directly usable in our model. Nominal CMT rates are par coupon rates and must be transformed into discount factor values. Nominal CMT rates are par coupon rates defined as follows:

(27)  \[
1 = \sum_{j=1}^{n(k)} C^i P_n(0, j\Delta) + P_n(n\Delta)
\]

where \( C^i \) is \( k \) -maturity coupon rate which is also the CMT rate and \( n(k) \) is the \( k \)-th bond’s number of coupons; for example, for a 5-year CMT rate, \( n(5) = 10 \) (semi-annual coupons) and \( \Delta = 0.5 \). The values of the discount factors, \( P_n(t, T) \) is estimated by linearly interpolating the forward rates:

(28)  \[
P_n(0, T) = \exp \left( -\sum_{j=1}^{t} f_n^{(j)}(\tau_j) \right)
\]

where \( f_n^{(j)} \) is the piece-wise flat nominal forward rate over the period \( \tau_j \). The standard calculations of the discount factors using the CMT rates assume 3, 5, 10, and 20 years.

We use the similar approach as the real rate to estimate the inflation rate (implied from the nominal rate). In this step, we specify the state propagation equation using an Euler approximation of statistical dynamics of the inflation risk factor embedded in equation (3) as:

(29)  \[
X_t = \Phi X_t + \sqrt{Q} \varepsilon_t \quad X = \begin{pmatrix} x \ y \end{pmatrix}
\]

where \( \varepsilon_t \) are i.i.d. standard normal innovation and

(30)  \[
\Phi = \exp(-D\Delta t), \quad D = \begin{pmatrix} \sigma^2 / 2 & 0 \\ 0 & \rho g \sigma \end{pmatrix}, \quad Q = \begin{pmatrix} \rho g \sigma & \rho g \sigma \\ \rho g \sigma & g^2 / 2 \end{pmatrix} \Delta t \cdot \Delta t = 7 / 365 \text{ corresponds to weekly frequency of the data.}
\]

The measurement equations for the nominal rate are defined as:
\[(Y^k_n) = Y^k_n + \epsilon^k_n,\]

where, similarly defined as (25), \(Y^k_n(t) = -\frac{\ln(P_n(t, 1 + e_n))}{\tau} \) is the return of the nominal zero-coupon bond price from equation (20) and \(k\) represents various CMT maturities. We assume that all of the nominal rates are priced with measurement error and derive the conditional likelihood function of 10 nominal CMT series given the parameters of the real rate from step 1 as:

\[
\max L_n(\Psi, \{y_t\} | \Theta) = \sum_{t=1}^{N} \left[ \frac{1}{2} \log |V_t| - \frac{1}{2} \left( (y_t - \Psi)^T V_t^{-1} (y_t - \Psi) \right) \right]
\]

where \(\Psi = \{a, b, g, l, \rho\}, \Theta = \{\alpha, \mu, \sigma, \lambda\}\).

**Unscented Kalman Filter**

The Kalman filter and its various extensions belong to the state space estimation regime and are based on a pair of state propagation equations and measurement equations. In our applications with Gaussian state variables, we can write the state-propagation generally as:

\[(33) \quad X_t = A + \Phi X_{t-1} + \sqrt{Q} \epsilon_t\]

We can also write the measurement equation in a generic form,

\[(34) \quad Y_t = h(X_t; \Theta) + \epsilon_t\]

where \(Y_t\) (could be \(Y_t\) or \(Y^r_t\)) denotes the observed series at time \(t\) and \(h(X_t; \Theta)\) denotes their corresponding fair values based on the model, as a function of the state vector \(X_t\) and model parameters \(\Theta\). The last term \(\epsilon_t\) denotes the measurement error on the series at time \(t\).

Let \(\hat{X}_t, \hat{\Sigma}_t, \hat{V}_t, \hat{\psi}_t\) denote the time \(t - 1\) ex ante forecasts of time \(t\) values of the state vector, the covariance of the state vector, the measurement series, and the covariance of the measurement series. Let \(\hat{X}_t\) and \(\hat{\Sigma}_t\) denote the ex post update, or filtering, on the state vector and its covariance at the time \(t\) based on observations (\(Y_t\)) at time \(t\). In the case of linear measurement equation,

\[(35) \quad Y_t = HX_t + \epsilon_t\]

The Kalman filter provides the most of the efficient updates. The ex ante predictions are:

\[
\begin{align*}
\hat{X}_t &= A + \Phi \hat{X}_{t-1} \\
\hat{\Sigma}_t &= \Phi \hat{\Sigma}_{t-1} \Phi^T + Q \\
\hat{V}_t &= H \hat{X}_t \\
\hat{\psi}_t &= H \hat{\Sigma}_t H^T + R
\end{align*}
\]

and the ex post filtering updates are:
\[
\dot{X}_{t+1} = \dot{X}_{t+1} + K_{t+1}(Y_{t+1} - \overline{Y}_{t+1}) \\
\dot{\Sigma}_{t+1} = \Sigma_{t+1} - K_{t+1}V_{t+1}K_{t+1}^T
\]

where \( K_{t+1} \) is the Kalman gain, given by:

\[
K_{t+1} = \Sigma_{t+1}H^T(\overline{V}_{t+1})^{-1}
\]

However, in our application, the measurement equation in equation (34) is nonlinear.
Traditionally, nonlinearity is often handled by the extended Kalman filter (EKF), which approximates the nonlinear measurement equation with a linear expansion, evaluated at the predicted states:

\[
Y_i \approx H(\overline{X}_i; \Theta)X_i + e_i
\]

where

\[
H(\overline{X}_i; \Theta) = \left. \frac{\partial h(\overline{X}_i; \Theta)}{\partial X_i} \right|_{X_i = \overline{X}_i}
\]

The rest of prediction and updates follow equation (35) and (36). We note that the extended Kalman filter uses only one point (the conditional mean) from the prior filtering density for the prediction and filtering updates.

In contrast, the unscented Kalman filter applied in this paper uses a set of points that are designed to also match higher moments. Let \( p = 3 \) be the number of states and \( \delta > 0 \) be a control parameter. Let \( A_i \) be the \( i \)-th column of a matrix \( A \). A set of \( 2p + 1 \) sigma vectors \( \chi_i \) are generated according to the following equations:

\[
\chi_{0,i} = \dot{X}_i \\
\chi_{i,i} = \dot{X}_i \pm \sqrt{(p + \delta)(\dot{\Sigma}_i + Q_j)}, j = 1,...,p; i = 1,...,2p
\]

with corresponding weights \( w_i \) given by

\[
w_0 = \frac{\delta}{(p + \delta)} \quad w_i = \frac{1}{(2(p + \delta))}, \quad i = 1,...,2p
\]

We can regard these sigma vectors as forming a discrete distribution with \( w_i \) as the corresponding probabilities. Then we can verify that the mean, covariance, skewness, and kurtosis of this distribution are \( \dot{X}_i, \dot{\Sigma}_i + Q, 0 \), and \( p + \delta \), respectively. Given the sigma points, the prediction steps are given by
\[
\hat{X}_{t+1} = \sum_{i=0}^{2\mu} w_i (\Phi \chi_{t,i}) \\
\bar{\Sigma}_{t+1} = \sum_{i=0}^{2\mu} w_i (\Phi \chi_{t,i} - \hat{X}_{t+1}) (\Phi \chi_{t,i} - \hat{X}_{t+1})^T \\
\bar{Y}_{t+1} = \sum_{i=0}^{2\mu} w_i h (\Phi \chi_{t,i}; \Theta) \\
\bar{V}_{t+1} = \sum_{i=0}^{2\mu} w_i \left( h (\Phi \chi_{t,i}; \Theta) - \bar{Y}_{t+1} \right) \left( h (\Phi \chi_{t,i}; \Theta) - \bar{Y}_{t+1} \right)^T + R
\]

and the filtering updates are given by
\[
\hat{X}_{t+1} = \bar{X}_{t+1} + K_{t+1}(Y_{t+1} - \bar{Y}_{t+1}) \\
\hat{\Sigma}_{t+1} = \bar{\Sigma}_{t+1} - K_{t+1}\bar{V}_{t+1}K_{t+1}^T
\]

with the Kalman gain defined as
\[
K_{t+1} = \bar{S}_{t+1} \left( \bar{V}_{t+1} \right)^{-1} = \sum_{i=0}^{2\mu} w_i (\Phi \chi_{t,i} - \bar{X}_{t+1}) \left[ h (\Phi \chi_{t,i}) - \bar{Y}_{t+1} \right] \left[ h (\Phi \chi_{t,i}) - \bar{Y}_{t+1} \right]^T \left( \bar{V}_{t+1} \right)^{-1}
\]

\section{Empirical Results}

\subsection*{Data}

Two datasets are used in this study. The first dataset is a weekly dataset of constant maturity treasury (CMT) rates of ¼, ½, 1, 2, 3, 5, 10, 20, and 30 years to maturity from the same period, which is downloaded from the website of St. Louis Federal Reserve Board

\url{http://research.stlouisfed.org/fred2}.

The second dataset includes weekly prices of the entire collection of the TIPS since its first appearance in January 1998 till October 2007 from Bloomberg. In Table 2, we list all existing TIPS from 1997. Up to October 2007, the U.S. Treasury department has issued 25 TIPS. Among these TIPS, two have expired. We use all of these TIPS in our estimation except the first issued one.\textsuperscript{12} Most of these TIPS are issued in maturity of 5, 10, 20 and 30 years, and the semiannual interest payments are based on the inflation-adjusted principal at the times when the interests are paid. We obtain the weekly TIPS price data (every Friday’s closing price, if there is no Friday’s data, we use the previous Thursday’s data) from January 1998 through October 2007. Table 3 describes some summary statistics of TIPS and nominal CMTs. Figure 3 shows the average yields, the one standard deviation bounds of 13 TIPS traded in the market from January 1998 to October 2007. Note that at the beginning there were very few TIPS. The number of bonds is plotted on the right axis. Figure 3 confirms the observation in Figure 2 that the largest

\textsuperscript{12} There was one TIPS (CUSIP 9128273A8) issued on 7/15/1997, which has expired on 7/15/2002. Similar to Jarrow and Yildirim (2003), we exclude this bond in our analysis.
variation among TIPS yields should coincide with the steepest term structure estimated, which is in the period of 2002–2004.

The index for measuring the inflation rate is the non-seasonally adjusted U.S. City Average All Items Consumer Price Index for All Urban Consumers (CPI-U), published monthly by the Bureau of Labor Statistics (BLS), lagged by two months. At maturity, the securities will be redeemed at the greater of their inflation-adjusted principal or par amount at original issue\textsuperscript{13}.

\textbf{Estimation Results}

As a quick diagnosis, we perform the principal component analysis (PCA) with the real yields from the TIPS. Given that TIPS, like Treasuries, have fixed \textit{maturities} and will “roll down” as time passes, they cannot be matched with CMT rates that maintain a fixed \textit{time to maturity}, in order to perform the PCA, We use (23) to compute “constant maturity” real rates from the TIPS prices. Besides constant maturity real rate implied in TIPS, nominal CMT rates from 3m to 30y have also been used for the PCA.

Figure 4 shows strong evidence of the existence of one strong common factor in the real rates. The first common factor explains 97.26\% of the variation while the remaining three factors only explain less than 3\% of the variation. This result supports a one-factor model for the real rates. When we look at the PCA of the \textit{nominal CMT}, there exist strong evidence to include two factors to explain nominal rate, which is also consistent with previous literature of multifactor explanation for nominal interest rate. This result supports the use of a one-factor model for the inflation rate factor that we shall estimate with our model.

The estimation results of the real interest rate process using the TIPS data and of the inflation factor process using the CMT data are presented in Table 4. All of the parameters are reasonable and statistically significant. In addition to the parameters, the estimation process also computes time series estimates of the instantaneous real rate and the instantaneous inflation rate factor, which are plotted in Figure 5.\textsuperscript{14} The sum of the two is the instantaneous nominal rate. Figure 6 compares the estimated inflation rate factor that is implied by the market and the historical inflation rate from CPI-U. The estimated series that reflect expected inflation is much smoother than the realized while the general pattern is the same. This indicates that the realized series contains large amount of noise.

\textsuperscript{13} Hence the price of TIPS includes a put option price even though the put option normally has little change to be executed.

\textsuperscript{14} Note that instantaneous rates do not carry premia. This is not to be confused with term rates (e.g. 5-year nominal rate) that do.
There is a clear positive correlation between the real and inflation rates while the pattern is reversed toward the end of the period. The implied correlation between the real rate and the inflation rate is 0.6, which is contrary to the Mundell-Tobin effect and different from that was estimated in Jarrow and Yildirim (2003).

In order to gauge the pricing performance of our model, we compute the root mean square errors (RMSE) between the model prices and the actual prices. For each day, we compute TIPS prices and CMT rates with the parameters estimated in Table 4 and the formulas of (21) and (8) for the TIPS and (27) and (20) for the CMT rates. The results are reported in Table 6.

The mean errors of TIPS vary. The lowest mean error in absolute magnitude is –17 basis points (TIPS 10) and the highest is –1.20% (TIPS 14). The average is –33 basis points. The root mean squared errors (RMSE) also vary. The lowest is 28 basis points (TIPS 24) and the highest is 2.54% (TIPS 8). The average RMSE is $1.14 or 1.07% of the market value. Our results are close to the Chen-Scott (1993) two-factor model (1.35%) but significantly larger than the Brown and Shafer (1986) cross-sectional estimation of 1.7 cents.

CMT rates are percentage quotes already hence we do not compute percentage errors. The average mean error is –24 basis points and the average RMSE is 74 basis points. Again, this is roughly the same as the Chen-Scott two-factor result (47 basis points).

**Term Structure of Inflation Risk Premia**

Early evidence on the inflation risk premium was predominantly found in the U.K. Campbell and Shiller (1996) were the first to estimate the inflation risk premium. They use the index link gilts over the period of 1985 through 1994 in the U.K. and find that the inflation risk premium to be 70 to 100 basis points. However, their definition of the inflation risk premium is proxied by the excess return of a long-term bond over a short-term bond, which incorporates other important term premia such as liquidity preference and term risk premia. Due to the lack of a true inflation-indexed asset, they construct a hypothetical indexed bond with real interest rates and inflation. Using the inflation-indexed bonds in the U.K., Evan (1998) confirms the finding of the inflation risk premium by Campbell and Shiller. Using the survey data and the data by the monetary authorities, Shen (1998) finds the inflation risk premium to be similar to the Campbell-Shiller result, while the survey data show a slightly lower inflation risk premium.

Before TIPS were available, due to the lack of market data, the estimation of inflation risk premium in the U.S. must rely upon macroeconomic data such price index (e.g. CPI-U) or money supply. Using a structural model, Buraschi and Jiltsov (2005) relate, in a closed-end form, inflation risk premia with money supply and real productivity. However, their empirical analysis
is based on the information of nominal rate, price level and money supply. Using the CPI-U as a proxy for the inflation, Ang and Bekaert (2004) derive inflation risk premium within a regime shift framework. Yet their model induces that the inflation risk premium will reduce to zero when the correlation between real rate and inflation is zero, which is far from our results. Jarrow and Yildirim (2003) were the first to use the market information (TIPS) to estimate the HJM term structure model but due to the limitation of the HJM-style models, no inflation risk premium can be estimated in their study.

Our two-factor model hence is the first study to use the market price to estimate the inflation risk premium in the U.S. We find very different inflation risk premia than those by Ang and Bekaert (2004) and Buraschi and Jiltsov (2005). Our inflation risk premia are substantially lower than those found in Ang and Bekaert (2004) and Buraschi and Jiltsov (2005), while they still present a positive slope term structure. Table 7 summarizes the studies on inflation risk premium.

In the previous section we use UKF with TIPS and nominal CMT to estimate the parameters of the inflation dynamics in our model. In particular, we estimate the market price of the inflation risk. In order to estimate the inflation risk premium, we use a simple way by comparing the market price of zero-coupon bond with risk-neutral parameters and real parameters, i.e.,

\[
\text{IRP}(t, t + \tau) = P_n(t, t + \tau | \ell = 0, \Theta) - P_n(t, t + \tau | \ell = \ell', \Theta)
\]

where \( IRP(t, t + \tau) \) is the inflation risk premium of maturity \( \tau \), \( P_n(t, t + \tau | \ell = 0) \) and \( P_n(t, t + \tau | \ell = \ell') \) are model prices of nominal zero-coupon bond given by equation (20)\textsuperscript{15}. For our sample period, we compute \( \text{IRP}(t, t + \tau) \) for \( T = \frac{1}{4}, \frac{1}{2}, 1, 2, 5, 7, 10, 20 \) and 30 years and plot the results in Panel A of Figure 7. The term structure of the inflation risk premia is positive sloped.

From Panel B of Figure 7, we find the average inflation risk premium for 3-month, 6-month, 1-year, 2-year, 5-year, 7-year, 10-year, and 20-year, maturity bond to be 0.24, 0.65, 1.95, 6.19, 12.01, 25.69, 39.15, 19.03, 25.69 and 77.24 basis points respectively during our sample period from January 1998 to October 2007. Compared to the inflation rate illustrated in Figure 5, the inflation risk premia are quite stable, especially for the short maturity bonds. Consistent with Buraschi and Jiltsov (2005), the inflation risk premium has an obviously positive correlation with maturity of bonds. However, the range of our estimation for different maturity’s inflation risk premia is larger than Buraschi and Jiltsov (2005). For short maturity bonds, the inflation risk

\textsuperscript{15} Following CIR, \( a = \dot{a} + l, \ am = \dot{a} \dot{m} \) is applied to incorporate market price of risk into equation (20).
premia is nearly negligible, while 20-year maturity bond’s inflation risk premia is more than 77 basis points. From Figure 7, we also find that inflation risk premia have presented a hump during 2000 and increase gradually since 2003. The finding of substantially lower inflation risk premia for the period 2001-2004 also reflects the existence of much lower inflation risk during that period, as shown in Figure 6.

Test of the Fisher Equation

The Fisher approximation specifies that the nominal interest rate equals the sum of the real interest rate, the rate of inflation, and if any inflation risk premium. When the real rate of interest and the rate of inflation are both stochastic, the Fisher equation implies that they are independent. As pointed out in Ang and Bekaert (2003) and Buraschi and Jiltsov (2005), estimates of inflation risk premium for bonds maturing during the next five to ten years are surprisingly large, generally in a range between 35 and 100 basis points, depending on the time period studied. This is not a trivial quantity compared to the level of spread. In addition, the inflation risk premium seems to vary over time. This directly refutes the practical view that the difference between the nominal and real rates represents expected inflation.

Our results show that the correlation between the instantaneous real rate and the instantaneous inflation is high (see Table 4). Table 5 presents the summary statistics of the difference between the real and nominal rates for various maturities (spread) plotted in Figure 2. We find the spreads to be quite stable around 2% across various maturities. As indicated in Figure 2, these spreads are highly correlated. Theoretically, these spreads should reflect the expected inflation as well as various inflation risk premia. To test the Fisher equation in our model, we compute the difference between the spread and the sum of estimated constant maturity inflation rates (CMI) and inflation risk premium (IRP). The result is plotted in Figure 8.

\[
(46) \quad CMI(t, \tau) = -\frac{\ln(P(t, t + \tau))}{\tau}
\]

where \( P(t, t + \tau) = F(t, t + \tau)e^{-\int_l^t G(t, t + \tau)} \)

\[
F(t, t + \tau) = \left[ \frac{2\xi e^{(a+\xi)\tau/2}}{(a + \xi)(e^{\xi\tau} - 1) + 2\xi} \right]^{4\tau/\rho}, \quad G(0, T) = \frac{2(e^{\xi\tau} - 1)}{(a + \xi)(e^{\xi\tau} - 1) + 2\xi}
\]

\[\xi = \sqrt{a^2 + 2g^2}, \quad \iota(t) \text{ is instantaneous inflation rate} \]

As we can see, the errors are larger for longer maturities. For short maturity rates, there is an upward bias during the period between 2004 and 2006. On average, this error is negligible (mean error for 3-year and 5-year are 0.00196 and -0.00092 respectively), which supports the Fisher hypothesis that nominal rate is composed of real rate, expected inflation and compensation for inflation risk premium. However, there is a systematic downward bias for long maturity rates.
The model computed nominal rates from Fisher Equation (real interest rate, expected inflation rate and inflation risk premium) are consistently lower than the observed nominal rates (mean error for 10-year and 20-year are −0.00538 and −0.00478 respectively).

There are several possible explanations of the discrepancies. The obvious one is the liquidity premium. Although the number of TIPS has increased significantly, the outstanding quantity still remains small compared to the traditional Treasury securities. Moreover, TIPS are attractive to buy-and-hold investors, in contrast to traditional Treasury securities, which are extensively used for dynamic hedging. Therefore, the market of the TIPS remains less liquid than that for the conventional treasury securities. One possible partial fix for the liquidity problem is to use nominal yield of off-the-run securities because the off-the-run securities are less liquid than on-the-run (newly issued) securities.

**Correlation Analysis**

One contribution of this paper, which is unique in the literature, is that we can econometrically endogenously estimate the correlation between the instantaneous inflation rate and instantaneous real rate. This is due to the factor structure of the CIR model and our closed form solution provides easy estimation of this parameter. From Table 4 we find the estimated correlation between the instantaneous real rate and the instantaneous inflation rate is 0.5 and significant. Sun (1992) and Gibbons and Ramaswamy (1993) also consider the correlation between the real rate and the inflation rate, but they only focus on the time series correlation and not imply it from the market price. Brennan, Yang, and Xia (2004) develop an inter-temporal model for the nominal bond and build in the correlation between the market portfolio and the real rate. Barr and Campbell (1997) use UK interest rates and find that the unconditional correlation between real rates and inflation is small but positive, whereas the correlation between the change in the real rate and changes in expected inflation is strongly negative. Jarrow and Yildirim (2003) compute correlation historically. Ang and Bekaret (2004) find that correlations to be negative for the short dated yields and positive for the long dated yields.

Figure 9 presents various sensitivity analyses with respect to correlation. The first panel is the sensitivity of the likelihood ratio with respect to correlation, the second panel is sensitivity of the root mean squared error, and the last is the sensitivity of the inflation risk premium. When all other parameters are fixed at the optimal levels, correlation plays an important role in the pricing of both interest rate and inflation risk premia. Figure 1 already shows us the existence of highly correlation between inflation rate and interest rate. In Table 8 we display the correlation between the model implied real rate, the nominal rate and the inflation risk premia. Contrary to Buraschi and Jiltsov (2005), we obtain highly positive correlation between real rate and nominal rate.
Short-end maturity inflation risk premia are positively correlated with both real rate and nominal rate, while longer maturity inflation risk premia are negatively correlated with interest rate. Goto and Torous (2002) support our positive correlation result.

**VI Conclusion**

In this paper, we study inflation and the term structure of inflation risk premia using TIPS and a two-factor CIR term structure model for correlated real rates and inflation rates. The principle component analysis supports the two-factor model we use in the paper. The two-factor CIR model has a closed-form solution to both real and nominal bond prices. The analytical solution facilitates the estimation of model parameters with improved efficiency and accuracy.

The estimation result points out that the real rate process presents quick mean reversion (\( \hat{\alpha} = 0.73 \) or 0.95 years expected half mean life), and low volatility and the inflation rate process presents lower mean reversion (\( \hat{\alpha} = 0.413 \) or 1.67 years expected half mean life). The estimation result also shows that the implied correlation between the real rate and the inflation rate is positive, which confirms the result of Goto and Torous (2002) but is not consistent with the Mundell-Tobin effect.

The inflation risk premia are quite stable for all maturities. The term structure is positively sloped. The average inflation risk premia for different maturity bonds change from 0.24 basis points of 3-month bond to 77.24 basis points of 20-year bond.

The model supports the Fisher hypothesis that the nominal rate consists of real rate, expected inflation, and inflation risk premium. However, there exist a systematic upward bias for long maturity rates. This could be due to a number of reasons. The first is liquidity premium since TIPS are traded less liquidly than traditional Treasuries. Secondly, we approximate the real rate by TIPS yield. Thirdly, we apply a piece-wise flat forward curve to approximate the real yield curve from various-maturity TIPS. Fourthly, our model assumes constant risk premium. To examine these assumptions is left to future research.

**References**


**Appendix**

**Extension to multiple factors**

The extension to multiple factors is straightforward. To facilitate the exposition, we use the following matrix notation:

\[
Y(t) = \begin{bmatrix}
  y_1(t) \\
  \vdots \\
  y_n(t)
\end{bmatrix}
\]

\[
Z(t) = \begin{bmatrix}
  Z_1(t) \\
  \vdots \\
  Z_n(t)
\end{bmatrix}
\]
\[ \Gamma(t) = \begin{bmatrix} \gamma_{11}(t) & \cdots & \gamma_{1,n}(t) \\ \vdots & \ddots & \vdots \\ \gamma_{n,1}(t) & \cdots & \gamma_{n,n}(t) \end{bmatrix} \]

\[ A = \begin{bmatrix} -\alpha_t & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & -\alpha_s \end{bmatrix} \]

We then have:

\[
P(0, t) = E \left[ \exp \left( -\int_0^t r(u)du \right) \right] \\
= E \left[ \exp \left( -\int_0^t Y'(u)Y(u)du \right) \right] \\
= E \left[ \frac{d\mathbb{P}}{d\mathbb{P}} \exp \left( -\int_0^t Y'(u)Y(u)du \right) \right]
\]

where

\[
\frac{d\mathbb{P}}{d\mathbb{P}} = \exp \left( -\int_0^t Y'(t)\Sigma dZ(t) - \frac{1}{2} \int_0^t Y'(t)\Sigma \Sigma' \Gamma'(t)Y(t)dt \right)
= \exp \left( -\int_0^t \left( Y'(t)\Gamma(t)dY(t) - Y'(t)\Gamma(t)[A Y(t) + B(t)]dt - \frac{1}{2} \int_0^t Y'(t)\Sigma \Sigma' \Gamma'(t)Y(t)dt \right) \right)
\]

Using Ito on the first term of the last line of Eq. [1], we get,

\[
d[Y'(t)\Gamma(t)Y(t)] = Y'(t)\Gamma(t)dY(t) + d[Y'(t)\Gamma(t)Y(t)] + Y'(t) \frac{d\Gamma(t)}{dt} Y(t)dt \\
+ d[Y'(t)\Gamma(t)dY(t)] + Y'(t)\Gamma'(t)[dt]d[Y(t)] + d[Y'(t)\Gamma(t)dt Y(t)] \\
= 2Y'(t)\Gamma(t)dY(t) + Y'(t)\Gamma'(t)Y(t)dt + tr[\Sigma' \Gamma(t)\Sigma]dt
\]

Therefore

\[
Y'(t)\Gamma(t)dY(t) = \frac{1}{2} d[Y'(t)\Gamma(t)Y(t)] - \frac{1}{2} Y'(t)\Gamma'(t)Y(t)dt - \frac{1}{2} \text{tr}[\Sigma' \Gamma(t)\Sigma]dt 
\]

Substituting [2] back into [1], we get,

\[
\frac{d\mathbb{P}}{d\mathbb{P}} = \exp \left[ -\frac{1}{2} Y'(\tau)\Gamma(\tau)Y(\tau) + \frac{1}{2} Y'(0)\Gamma(0)Y(0) \\
+ \int_0^\tau \frac{1}{2} Y'(t) \frac{d\Gamma(t)}{dt} Y(t)dt + \int_0^\tau \frac{1}{2} \text{tr}[\Sigma' \Gamma(t)\Sigma]dt \\
+ \int_0^\tau Y'(t)\Gamma(t)A Y(t)dt - \frac{1}{2} \int_0^\tau Y'(t)\Gamma(t)\Sigma \Sigma' \Gamma'(t)Y(t)dt \right]
\]

Substituting the above result into Eq. [1],
Let the coefficient of square term is equal to zero.

\[ \frac{d\Gamma(u)}{du} + \Gamma(u)A + AI(u) - \Gamma(u)\Sigma t\Gamma(u) + 2I = 0 \]

where \( I \) is an identity matrix. Equation [5] is known as the Riccati equation. Define the terminal condition \( \Gamma(t) = 0 \) and let:

\[
N(u) = \begin{bmatrix} n_1(u-t) & n_2(u-t) \\ n_3(u-t) & n_4(u-t) \end{bmatrix} = \exp\left[ \begin{bmatrix} -A & -2I \\ -\Sigma t & A \end{bmatrix} (u-t) \right]
\]

Then the solution to Eq. [5] is:

\[ \Gamma(u,t) = n_2(u-t)[n_4(u-t)]^{-1} \]

Using the result of Eq. [6], Eq. [4] can be simplified to the following equation,

\[
P(0,t) = E \left[ \exp\left(-\int_0^t r(u)du\right) \right]
\]

\[ = E \left[ \exp\left( - \int_0^t Y'(u)Y(u)du \right) \right]
\]

\[ = E \left[ \frac{dP}{d\Phi} \exp\left( - \int_0^t Y'(u)Y(u)du \right) \right]
\]

\[ = \exp\left\{ - \frac{1}{2} \int_0^t Y'(0)\Gamma(0,t)Y(0) + tr[\Sigma t\Gamma(t)\Sigma]dt \right\}
\]
Solution to $\Xi$

\[ A = \sqrt{g^2(1 - \rho^2)} = g(1 - \rho^2) \]
\[ B = \sqrt{a^2 - g^2(1 - \rho^2)} = \sqrt{a^2 - A} \]
\[ C = \tanh^{-1} \left( \frac{-\frac{1}{2}a^2 + \frac{1}{4}A(1 + e^{2\tau A})}{aB} \right) \]
\[ x = a^2 - \sigma^2 \]

\[
\frac{1}{4}g^2\rho^2 \left( \frac{1}{4} \tau A(a - B) - \frac{1}{4}AC}{B} + \frac{1}{4}A \tanh^{-1} \left( \frac{1}{4} - \frac{1}{4} A \ln[-B^2] + \frac{1}{4} A \ln[-a^2 e^{-B} + \frac{1}{4} A(1 + e^{-B})^2] \right) \right)
\]

\[
\frac{1}{4}A(B(a - B) - \frac{1}{4} aAB^2 \tanh^{-1} \left( \frac{1}{4}(-a^2 + \frac{1}{2}A(1 + e^{-B}))}{aB} \right)}{\frac{1}{2}aB^2} + \frac{1}{4}AaB \tanh^{-1} \left( \frac{-A}{aB} \right) + \frac{1}{4} A \ln[-B^2] + \frac{1}{4} A \ln[-e^{-B}a^2 + \frac{1}{4} A(1 + e^{-B})^2] \right)
\]

\[
\frac{1}{A^2} \left( \frac{-\frac{1}{2} x \tanh^{-1} \left( \frac{\sqrt{x}}{a} \right) + \frac{1}{2} x \tan^{-1} \left( \frac{a^2 + \frac{1}{2}(1 + e^{-\sqrt{x}})\sigma^2}{\alpha \sqrt{x}} \right) + \sqrt{x} \left( \frac{-\frac{1}{2} \alpha \tau x + \frac{1}{4} \alpha \tau \sqrt{x} - \frac{1}{4} \alpha \sqrt{x} \ln[x] + \frac{1}{2} \alpha \sqrt{x} \ln \left[ \frac{1}{4}(1 + e^{-\sqrt{x}})^2 \right] \right)}{\frac{1}{4}x(a^2 - \frac{1}{4} \sigma^2)^2} \right)
\]
Table 1 International Markets of Inflation-indexed Debts as of December 2004

This table shows international issuance of inflation indexed public debts from 30 countries through December 2004. The data source comes from Bank of International Settlements and Bloomberg. We do not include debts that are indexed to other indexes.

<table>
<thead>
<tr>
<th>Country</th>
<th>First Year Issued</th>
<th>Outstanding Indexed Public Debt ($ billions)</th>
<th>Percent of Total Government Debt (%)</th>
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<td>Argentina</td>
<td>2002</td>
<td>35.64</td>
<td>41.15</td>
</tr>
<tr>
<td>Australia</td>
<td>1985</td>
<td>12.01</td>
<td>13.21</td>
</tr>
<tr>
<td>Austria</td>
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</tr>
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<td>1964</td>
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<td>28.78</td>
</tr>
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<td>1991</td>
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<td>4.22</td>
</tr>
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<td>1997</td>
<td>0.15</td>
<td>0.33</td>
</tr>
<tr>
<td>Denmark</td>
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<td>44.44</td>
</tr>
<tr>
<td>Finland</td>
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<td>0.01</td>
</tr>
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<td>France</td>
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<td>76.96</td>
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<td>2.15</td>
<td>0.21</td>
</tr>
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<td>Greece</td>
<td>1997</td>
<td>3.44</td>
<td>2.00</td>
</tr>
<tr>
<td>Hungary</td>
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</tr>
<tr>
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<td>3.50</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>42.08</td>
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</tr>
<tr>
<td>Japan</td>
<td>2004</td>
<td>399.3</td>
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<td>Mexico</td>
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<td>55.07</td>
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</tr>
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</table>
This table includes all TIPS issued through Oct. 2007 in the U.S., only one of which has expired. Our data sample ranges from January 1998 to Oct. 2007.

<table>
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<th>Coupon (%)</th>
<th>Maturity</th>
<th>Expiration Date</th>
<th>Issue Amt (bln)</th>
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Table 3 Summary Statistics on Yields of TIPS and Nominal CMTs
This table gives the summary statistics of TIPS yields and nominal CMT from January 1998 to October 2007.

<table>
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<th></th>
<th>Mean</th>
<th>Std</th>
<th>Min</th>
<th>Max</th>
<th>Obs (wks)</th>
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<td>0.00875</td>
<td>0.028899</td>
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<tr>
<td>TIPS16</td>
<td>0.021564</td>
<td>0.00315</td>
<td>0.01479</td>
<td>0.027496</td>
<td>146</td>
</tr>
<tr>
<td>TIPS17</td>
<td>0.022519</td>
<td>0.002375</td>
<td>0.016426</td>
<td>0.02726</td>
<td>120</td>
</tr>
<tr>
<td>TIPS18</td>
<td>0.023721</td>
<td>0.001671</td>
<td>0.019771</td>
<td>0.027253</td>
<td>93</td>
</tr>
<tr>
<td>TIPS19</td>
<td>0.023522</td>
<td>0.001785</td>
<td>0.019395</td>
<td>0.027372</td>
<td>94</td>
</tr>
<tr>
<td>TIPS20</td>
<td>0.023659</td>
<td>0.002116</td>
<td>0.018342</td>
<td>0.027631</td>
<td>80</td>
</tr>
<tr>
<td>TIPS21</td>
<td>0.023477</td>
<td>0.001595</td>
<td>0.02021</td>
<td>0.027185</td>
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</tr>
<tr>
<td>TIPS22</td>
<td>0.024151</td>
<td>0.001521</td>
<td>0.021556</td>
<td>0.027174</td>
<td>41</td>
</tr>
<tr>
<td>TIPS23</td>
<td>0.02383</td>
<td>0.001909</td>
<td>0.020536</td>
<td>0.02744</td>
<td>42</td>
</tr>
<tr>
<td>TIPS24</td>
<td>0.023828</td>
<td>0.002663</td>
<td>0.018647</td>
<td>0.027741</td>
<td>28</td>
</tr>
<tr>
<td><strong>Nominal CMT</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M3</td>
<td>0.035808</td>
<td>0.01727</td>
<td>0.0087</td>
<td>0.0641</td>
<td>507</td>
</tr>
<tr>
<td>M6</td>
<td>0.037149</td>
<td>0.017413</td>
<td>0.0089</td>
<td>0.0644</td>
<td>507</td>
</tr>
<tr>
<td>Y1</td>
<td>0.038045</td>
<td>0.016331</td>
<td>0.0096</td>
<td>0.0642</td>
<td>507</td>
</tr>
<tr>
<td>Y2</td>
<td>0.040526</td>
<td>0.014792</td>
<td>0.0116</td>
<td>0.069</td>
<td>507</td>
</tr>
<tr>
<td>Y3</td>
<td>0.042152</td>
<td>0.013074</td>
<td>0.0142</td>
<td>0.0684</td>
<td>507</td>
</tr>
<tr>
<td>Y5</td>
<td>0.045014</td>
<td>0.010259</td>
<td>0.0216</td>
<td>0.0677</td>
<td>507</td>
</tr>
<tr>
<td>Y7</td>
<td>0.047416</td>
<td>0.008966</td>
<td>0.0272</td>
<td>0.0678</td>
<td>507</td>
</tr>
<tr>
<td>Y10</td>
<td>0.048692</td>
<td>0.007089</td>
<td>0.0321</td>
<td>0.0673</td>
<td>507</td>
</tr>
<tr>
<td>Y20</td>
<td>0.053887</td>
<td>0.006</td>
<td>0.0423</td>
<td>0.0697</td>
<td>507</td>
</tr>
<tr>
<td>Y30</td>
<td>0.054755</td>
<td>0.004934</td>
<td>0.0448</td>
<td>0.0672</td>
<td>301</td>
</tr>
</tbody>
</table>
Table 4 Maximum Likelihood Estimates of Model Parameters

This table presents the parameter estimates (the t-statistics are in parentheses; standard errors of parameters are calculated from the outer-product [BHHH] formula) that govern the factor dynamics on real interest rates $r(t)$ and nominal rates $R(t)$. We take the two-step estimation approach to estimate the parameters. In the first step, we use UKF with measurement error of 13 time series of prices of TIPS. In the second step, we use another UKF to estimate the parameters that govern the nominal rate (parameters of the inflation process). ¼, ½, 1, 2, 3, 5, 10, 20, and 30 years nominal CMT are used in this step. The estimation is based on 513 weekly observations from January 1998 to October 2007.

<table>
<thead>
<tr>
<th>Real Rate</th>
<th>Implied Inflation Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>0.729976 (5.928)</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.053546 (32.516)</td>
</tr>
<tr>
<td>$\hat{\beta}$</td>
<td>0.001343 (3.891)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>-0.730417 (-5.926)</td>
</tr>
<tr>
<td>$\hat{\rho}$</td>
<td>0.50007 (2.096)</td>
</tr>
</tbody>
</table>

No. of Obs. | 513 | No. of Obs. | 513
Table 5 Summary Statistics on Inflation Spreads from January 1998 through October 2007

The spread is usually perceived as the expected inflation. For example, Spread 3 is the difference between the 3-year nominal CMT and the interpolated constant maturity 3-year real rate.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std.Dev</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spread3</td>
<td>0.0188</td>
<td>0.0071</td>
<td>0.0002</td>
<td>0.0306</td>
</tr>
<tr>
<td>Spread5</td>
<td>0.0193</td>
<td>0.0056</td>
<td>0.0045</td>
<td>0.0296</td>
</tr>
<tr>
<td>Spread10</td>
<td>0.0201</td>
<td>0.0042</td>
<td>0.0089</td>
<td>0.0275</td>
</tr>
<tr>
<td>Spread20</td>
<td>0.0242</td>
<td>0.0031</td>
<td>0.0163</td>
<td>0.0298</td>
</tr>
</tbody>
</table>

Pearson Correlation Coefficients

<table>
<thead>
<tr>
<th></th>
<th>Spread3</th>
<th>Spread5</th>
<th>Spread10</th>
<th>Spread20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spread3</td>
<td>1</td>
<td>0.9047</td>
<td>0.8526</td>
<td>0.7971</td>
</tr>
<tr>
<td>Spread5</td>
<td></td>
<td>1</td>
<td>0.9689</td>
<td>0.9131</td>
</tr>
<tr>
<td>Spread10</td>
<td></td>
<td></td>
<td>1</td>
<td>0.9546</td>
</tr>
<tr>
<td>Spread20</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
Table 6 Summary Statistics on Pricing Errors

This table reports the summary statistics of the pricing errors of the UKF for TIPS and nominal CMT. We define the pricing error as the difference between the observed price (nominal rate) and the model-implied fair values. The column Mean is the sample mean of the pricing errors; Std is the standard deviation; Max is the maximum absolute error; VR is the explained percentage variance, defined as one minus the ratio of pricing error variance to observed values' variance; RMSE is the root mean squared errors, and Error (%) is the percentage of the square root of MSE divided by the average observed values.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std</th>
<th>Max</th>
<th>VR</th>
<th>RMSE</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIPS1</td>
<td>1.0036</td>
<td>1.9013</td>
<td>5.7489</td>
<td>84.02%</td>
<td>2.0452</td>
<td>1.99%</td>
</tr>
<tr>
<td>TIPS2</td>
<td>0.8155</td>
<td>1.8200</td>
<td>2.6292</td>
<td>86.89%</td>
<td>1.9908</td>
<td>1.91%</td>
</tr>
<tr>
<td>TIPS3</td>
<td>-0.4168</td>
<td>1.9695</td>
<td>4.1445</td>
<td>97.88%</td>
<td>1.9836</td>
<td>1.76%</td>
</tr>
<tr>
<td>TIPS4</td>
<td>0.7528</td>
<td>1.7514</td>
<td>2.3045</td>
<td>89.42%</td>
<td>1.8035</td>
<td>1.69%</td>
</tr>
<tr>
<td>TIPS5</td>
<td>-0.3658</td>
<td>2.1207</td>
<td>4.1942</td>
<td>97.71%</td>
<td>2.0066</td>
<td>1.68%</td>
</tr>
<tr>
<td>TIPS6</td>
<td>0.5804</td>
<td>1.7162</td>
<td>2.5098</td>
<td>89.69%</td>
<td>1.6119</td>
<td>1.46%</td>
</tr>
<tr>
<td>TIPS7</td>
<td>0.4392</td>
<td>1.6335</td>
<td>2.5085</td>
<td>87.70%</td>
<td>1.4073</td>
<td>1.30%</td>
</tr>
<tr>
<td>TIPS8</td>
<td>1.9001</td>
<td>2.6235</td>
<td>1.8125</td>
<td>93.07%</td>
<td>2.5397</td>
<td>2.10%</td>
</tr>
<tr>
<td>TIPS9</td>
<td>0.2386</td>
<td>1.6410</td>
<td>2.6084</td>
<td>84.75%</td>
<td>1.2724</td>
<td>1.16%</td>
</tr>
<tr>
<td>TIPS10</td>
<td>-0.1731</td>
<td>1.5312</td>
<td>2.6271</td>
<td>79.72%</td>
<td>1.0584</td>
<td>0.98%</td>
</tr>
<tr>
<td>TIPS11</td>
<td>-0.5957</td>
<td>1.3101</td>
<td>2.5773</td>
<td>73.35%</td>
<td>0.9513</td>
<td>0.95%</td>
</tr>
<tr>
<td>TIPS12</td>
<td>-1.0403</td>
<td>1.1562</td>
<td>2.7005</td>
<td>81.66%</td>
<td>0.9674</td>
<td>0.96%</td>
</tr>
<tr>
<td>TIPS13</td>
<td>-0.6204</td>
<td>1.3619</td>
<td>2.6914</td>
<td>89.04%</td>
<td>0.8619</td>
<td>0.83%</td>
</tr>
<tr>
<td>TIPS14</td>
<td>-1.1968</td>
<td>1.0356</td>
<td>2.6455</td>
<td>84.84%</td>
<td>0.9179</td>
<td>0.92%</td>
</tr>
<tr>
<td>TIPS15</td>
<td>-0.6533</td>
<td>1.2158</td>
<td>2.5572</td>
<td>44.76%</td>
<td>0.7617</td>
<td>0.79%</td>
</tr>
<tr>
<td>TIPS16</td>
<td>-1.5373</td>
<td>0.7512</td>
<td>2.6567</td>
<td>89.73%</td>
<td>0.9122</td>
<td>0.95%</td>
</tr>
<tr>
<td>TIPS17</td>
<td>-1.4917</td>
<td>0.7502</td>
<td>2.6641</td>
<td>84.51%</td>
<td>0.8069</td>
<td>0.83%</td>
</tr>
<tr>
<td>TIPS18</td>
<td>0.2176</td>
<td>1.1278</td>
<td>0.4803</td>
<td>82.65%</td>
<td>0.4865</td>
<td>0.51%</td>
</tr>
<tr>
<td>TIPS19</td>
<td>-1.5701</td>
<td>0.7912</td>
<td>2.6869</td>
<td>73.86%</td>
<td>0.7518</td>
<td>0.77%</td>
</tr>
<tr>
<td>TIPS20</td>
<td>-1.3233</td>
<td>0.6099</td>
<td>2.6563</td>
<td>62.42%</td>
<td>0.5748</td>
<td>0.57%</td>
</tr>
<tr>
<td>TIPS21</td>
<td>-1.2673</td>
<td>0.7737</td>
<td>2.2133</td>
<td>75.90%</td>
<td>0.5395</td>
<td>0.53%</td>
</tr>
<tr>
<td>TIPS22</td>
<td>0.6411</td>
<td>1.2918</td>
<td>0.1198</td>
<td>78.25%</td>
<td>0.4037</td>
<td>0.40%</td>
</tr>
<tr>
<td>TIPS23</td>
<td>-1.1686</td>
<td>0.8077</td>
<td>2.3930</td>
<td>80.79%</td>
<td>0.4049</td>
<td>0.40%</td>
</tr>
<tr>
<td>TIPS24</td>
<td>-1.0790</td>
<td>0.5913</td>
<td>1.7181</td>
<td>81.22%</td>
<td>0.2863</td>
<td>0.29%</td>
</tr>
<tr>
<td>Average</td>
<td>-0.3296</td>
<td>1.3451</td>
<td>2.5770</td>
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<td>1.1394</td>
<td>1.07%</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
<td>Max</td>
<td>VR</td>
<td>RMSE</td>
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<td>-------</td>
<td>------</td>
<td>------</td>
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</tr>
<tr>
<td>M3</td>
<td>-0.0065</td>
<td>0.0100</td>
<td>0.0302</td>
<td>66.31%</td>
<td>0.0119</td>
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<tr>
<td>M6</td>
<td>-0.0055</td>
<td>0.0102</td>
<td>0.0302</td>
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<td>0.0116</td>
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</tr>
<tr>
<td>Y1</td>
<td>-0.0053</td>
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<td>0.0288</td>
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<td>0.0108</td>
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<tr>
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<td>-0.0040</td>
<td>0.0082</td>
<td>0.0275</td>
<td>69.54%</td>
<td>0.0091</td>
<td></td>
</tr>
<tr>
<td>Y3</td>
<td>-0.0033</td>
<td>0.0069</td>
<td>0.0259</td>
<td>71.81%</td>
<td>0.0077</td>
<td></td>
</tr>
<tr>
<td>Y5</td>
<td>-0.0018</td>
<td>0.0052</td>
<td>0.0232</td>
<td>73.97%</td>
<td>0.0055</td>
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<td>0.0047</td>
<td>0.0222</td>
<td>72.65%</td>
<td>0.0047</td>
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<tr>
<td>Y10</td>
<td>-0.0002</td>
<td>0.0045</td>
<td>0.0201</td>
<td>59.91%</td>
<td>0.0045</td>
<td></td>
</tr>
<tr>
<td>Y20</td>
<td>0.0030</td>
<td>0.0045</td>
<td>0.0192</td>
<td>43.74%</td>
<td>0.0054</td>
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<tr>
<td>Y30</td>
<td>0.0002</td>
<td>0.0034</td>
<td>0.0159</td>
<td>52.83%</td>
<td>0.0026</td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>-0.0024</td>
<td>0.0067</td>
<td>0.0243</td>
<td>64.37%</td>
<td>0.0074</td>
<td></td>
</tr>
<tr>
<td>Study</td>
<td>IRP (base points)</td>
<td>Model</td>
<td>Data</td>
<td>Period</td>
<td>Estimation Method</td>
<td>Country</td>
</tr>
<tr>
<td>------------------------------</td>
<td>-------------------</td>
<td>------------------------</td>
<td>-----------------------------------------------------------------------</td>
<td>------------</td>
<td>------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>Campbell &amp; Shiller (1996)</td>
<td>70~100 on 5-year</td>
<td>CAPM</td>
<td>Interest rate (zero-coupon bond yield), market portfolio, aggregate consumption</td>
<td>1953~1994</td>
<td>Regression</td>
<td>U.S.</td>
</tr>
<tr>
<td>Shen (1998)</td>
<td>74<del>104 (survey), 133</del>164 (monetary authorities)</td>
<td>nominal rate = real rate + inflation + IRP</td>
<td>Real interest rate (index-linked gild), nominal interest rate, expected inflation rate (survey &amp; monetary authorities)</td>
<td>1996~1997</td>
<td>Difference between interest spread and expected inflation</td>
<td>U.K.</td>
</tr>
<tr>
<td>Buraschi &amp; Jiltsov (2005, JFE)</td>
<td>25 ~70</td>
<td>Structural model</td>
<td>Interest Rate (zero-coupon yield), inflation rate (CPI-U), money supply (M2)</td>
<td>1960~2000</td>
<td>Quasi-MLE</td>
<td>U.S.</td>
</tr>
<tr>
<td>Our model</td>
<td>1.35 ~ 22.31</td>
<td>Two-factor correlated CIR model</td>
<td>Real interest rate (TIPS), Nominal interest rate (CMT)</td>
<td>1998~2004</td>
<td>Unscented Kalman Filter</td>
<td>U.S.</td>
</tr>
</tbody>
</table>
Table 8 Correlations between Model Implied Series

This table presents the correlation matrix of model implied nominal rates, real rates, and inflation risk premia for maturity of 3y, 5y, 10y, and 20y. N3y, N5y, N10y, and N20y are fitted nominal rates from the model; R3y, R5y, R10y, and R20y represent real rates from TIPS; IRP3y, IRP5y, IRP10y, and IRP20y are calculated inflation risk premia respectively.

<table>
<thead>
<tr>
<th></th>
<th>Ny3</th>
<th>Ny5</th>
<th>Ny10</th>
<th>Ny20</th>
<th>Ry3</th>
<th>Ry5</th>
<th>Ry10</th>
<th>Ry20</th>
<th>IRPy3</th>
<th>IRPy5</th>
<th>IRPy10</th>
<th>IRPy20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ny3</td>
<td>1.000</td>
<td>0.984</td>
<td>0.895</td>
<td>0.681</td>
<td>0.853</td>
<td>0.823</td>
<td>0.713</td>
<td>0.533</td>
<td>0.465</td>
<td>0.345</td>
<td>-0.218</td>
<td>-0.563</td>
</tr>
<tr>
<td>Ny5</td>
<td>1.000</td>
<td>0.959</td>
<td>0.790</td>
<td>0.867</td>
<td>0.861</td>
<td>0.786</td>
<td>0.631</td>
<td>0.355</td>
<td>0.224</td>
<td>-0.343</td>
<td>-0.654</td>
<td></td>
</tr>
<tr>
<td>Ny10</td>
<td>1.000</td>
<td>0.923</td>
<td>0.836</td>
<td>0.873</td>
<td>0.860</td>
<td>0.762</td>
<td>0.159</td>
<td>0.014</td>
<td>-0.529</td>
<td>-0.768</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ny20</td>
<td>1.000</td>
<td>0.751</td>
<td>0.846</td>
<td>0.913</td>
<td>0.908</td>
<td>-0.104</td>
<td>-0.263</td>
<td>-0.760</td>
<td>-0.891</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ry3</td>
<td>1.000</td>
<td>0.966</td>
<td>0.899</td>
<td>0.776</td>
<td>0.249</td>
<td>0.090</td>
<td>-0.533</td>
<td>-0.820</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ry5</td>
<td>1.000</td>
<td>0.968</td>
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Figure 1 Correlation Between 1-year CMT and Inflation

This figure shows the monthly correlation between 1-year CMT and inflation rate using CPI-U from 1959 through 2007.
Figure 2 Nominal CMTs, Constant Maturity Real Rates, and their Differences

The following figures plot the time series of the 3-, 5-, 10-, and 20-year nominal CMTs, implied corresponding constant maturity real rates, and their differences, from January 1998 to October 2007. Nominal CMTs are downloaded from Federal Reserve and constant maturity real rates are calculated from TIPS prices by equation (23).
Figure 3 Yields of TIPS

This figure shows the average yields, the one standard deviation bounds of 13 TIPS traded in the market from January 1998 to October 2007. Note that at the beginning there were very few TIPS. The number of bonds is plotted on the right axis.
Figure 4 Principal Component Analysis of constant maturity real rates and Nominal CMTs
This graph shows the principal component analysis of 3-, 5-, 10-, and 20-year implied constant maturity real rates and 3m, 6m, 1y, 2y, 3y, 5y, 7y, 10y, 20y and 30y nominal CMTs. For nominal CMTs, only the first four components have been graphed. The data range from January 1999 to October 2007.
Figure 5  Estimated Instantaneous Real Rate and Inflation Rate

This figure shows the estimated instantaneous real rate (using TIPS) and inflation rates (using CMT) from January 1998 to October 2007. Nominal rate is the summation of the real spot rate and inflation spot rate.
Figure 6 Estimated Instantaneous Inflation Rate and CPI-U Historical Inflation Rate
This figure compares the annual real inflation rate calculated from CPI-U and the estimated inflation rate in the nominal CMT from January 1998 to October 2007.
This figure shows the time series of estimated inflation risk premia for 3-, 6-month, 1-, 2-, 3-, 5-, 7-, 10-, and 20-year maturity bonds from January 1998 to October 2007. The inflation risk premium is calculated as the difference of the model price of zero-coupon bond with risk neutral parameters $\ell = 0$ and the model price of zero-coupon bond with real parameters $\ell < 0$ in equation (45).

This figure shows the term structure of inflation risk premia for various maturities. It is simply takes averages of each of the above time series.
Figure 8 Test of Fisher Equation

This figure shows the difference (pricing error) between the nominal CMT rate and the Fisher equation which is the sum of the estimated real rate, inflation rate, and inflation risk premium. The estimated real rate series are the same as those used in PCA in Figure 4.
Figure 9  Sensitivity Analysis of Correlation

Panel A displays the log-likelihood sensitivity with $\rho$ (rho) for the second step estimation. All other parameters except $\rho$ are fixed at the optimal value. Panel B shows sensitivity of the average root mean square error of the second-step estimation error with the change of $\rho$. Panel C plots the average inflation risk premia sensitivity with $\rho$. 

![Loglikelihood Sensitivity with Rho](image)